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The Energy Dependence of Meson Multiplicity in the High-Energy Interactions.

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Summary. — An attempt has been made to analyse some experimental data of high-energy showers so far reported. It is shown that the dependence of the meson multiplicity on the primary energy seems to be explained phenomenologically, at least within the present limitation of experimental accuracy, in terms of the single nucleon-nucleon collision by making allowance for the fluctuation in inelasticity. The result seems to be consistent with the assumption that the average energy of produced mesons in the center-of-mass system is fairly constant (about 200 MeV kinetic energy) independently of the primary energy at least up to about 100 GeV (in the laboratory system) so that the meson multiplicity normalized at the value 1 of inelasticity increases rather sharply with the primary energy. At higher energies, the collimation of high-energy mesons into the forward and backward cones would result in increasing the average meson energy so that the increase of the normalized multiplicity with the primary energy seems to become slower than at lower energies. The results of the analysis are discussed in comparison with the existing theories of multiple meson production.

1. – Introduction.

There have been found by numerous workers the phenomena which could be explained in terms of single very high-energy nucleon-nucleon collisions in nuclear emulsions. Yet the plausible interpretation capable to apply to all the existing experiments is not established because of the experimental

inaccuracies and of the complexity of the phenomena. The dependence of meson multiplicity on the primary energy, for instance, could not yet be determined with a sufficient accuracy. As is well known, if we plotted the number of charged secondary particles against the estimated primary energy for these high-energy jets, the experimental points were distributed very widely. This fact has led some authors [1] to conclude that the observed jets consist of two groups with large and small inelasticities respectively. On the other hand, AMAI *et al.* [2] have pointed out recently the possibility that the apparent dispersion of experimental points might be explained in terms of a possible difference between the meson-nucleon and the nucleon-nucleon collisions.

In order to examine the energy dependence of meson multiplicity, however, it seems probable that we must take into account the fluctuation in inelasticity of individual events. Accordingly, by confining ourselves to those events which could be well explained by the single nucleon-nucleon interaction and by making allowance for the inelasticity of the event, we may obtain a satisfactory interpretation of the energy dependence of meson multiplicity. In particular, an inference from the knowledge of recent experimental results of the antiproton-nucleon annihilation process may make the situation at lower energies more plausible. With this aim we make an attempt to analyse some experimental data now existing.

2. – Analysis.

Although we are mainly interested in the single nucleon-nucleon interactions, we are forced to take those events which appear to be nucleon-nucleus collisions, because only a very few, well established examples of the elementary processes have been reported. Thus, we have selected the events satisfying all the following criteria:

i) The number of heavily ionizing prongs emerged from the shower is less than four. — This criterion assures, at least at relatively low energies (less than several hundred GeV), that the interaction has taken place probably between the incident nucleon and a hydrogen nucleus or a single nucleon bound at the periphery of a heavier nucleus. At extremely high-energies, however, this criterion does not exclude the possibility that the interaction has occurred in a central region of the target nucleus, since the incident nucleon may drive

(1) C. DILWORTH, S. J. GOLDSACK, T. F. HOANG and L. SCARSI: *Nuovo Cimento*, **10**, 1261 (1953); T. F. HOANG: *Journ. de Phys.*, **14**, 395 (1953); G. BERTOLINO and D. PESCETTI: *Nuovo Cimento*, **12**, 630 (1954).

(2) S. AMAI, T. MUROTA and M. NISHIDA: *Progr. Theor. Phys.*, **17**, 807 (1957). C. DILWORTH *et al.* (1), HOANG (1) and other authors have already suggested in a qualitative manner, the presence of this possibility.

a tunnel through the nucleus giving a very small excitation energy ([3]). But the present analysis is carried out tentatively on the assumption that this criterion ensures the event to be due mainly to a single collision between two nucleons. The event which bears some evidence for the existence of successive interactions of the primary particle inside the target nucleus, for instance a jet $(3+55_n)$ observed by LOHRMANN ([4]), has been rejected.

ii) At least for some of the secondary particles, the energies have been estimated on the basis of multiple scattering measurements or electron pair observations.

iii) The Lorentz transformation from the laboratory to the center-of-mass systems has been carried out by taking into account the angles and energies of secondary particles.

iv) The inelasticity of the interaction has been estimated from data ii) and iii). — The event in which the inelasticity was estimated simply from the maximum angle of shower particles has therefore been rejected.

In Table I are shown the events ([5-13]) thus selected for the present analysis. It should be noted that the primary particles of these cosmic-ray showers were altogether singly charged except for shower 12. The number of heavily ionizing (black and grey) prongs accompanied by the shower is given in column 4, which does not involve tracks identified as slow π-mesons. The multiplicity of charged mesons, $n_{\pm}$, shown in column 6 involves these grey and black π-mesons, whereas it does not the identified secondary protons of relativistic velocities. For the event which did not allow to identify shower particles, we

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([3]) F. C. ROESLER and C. B. A. MCCUSKER: *Nuovo Cimento*, **10**, 127 (1952); C. B. A. MCCUSKER and F. C. ROESLER: *Nuovo Cimento*, **5**, 1136 (1957).

([4]) E. LOHRMANN: *Zeits. f. Naturfor.*, **11a**, 561 (1956); *Nuovo Cimento*, **3**, 822 (1956).

([5]) W. H. BARKAS, R. W. BIRGE, W. W. CHUPP, A. G. EKSPONG, G. GOLDHABER, S. GOLDHABER, H. H. HECKMAN, D. H. PERKINS, J. SANDWEISS, E. SEGRÈ, F. M. SMITH, D. H. STORK, L. VAN ROSSUM, E. AMALDI, G. BARONI, C. CASTAGNOLI, C. FRANZINETTI and A. MANFREDINI: *Phys. Rev.*, **105**, 1037 (1957).

([6]) S. KANEKO, O. KUSUMOTO and S. MATSUMOTO: *Journ. Phys. Soc. Jap.*, **11**, 1129 (1956).

([7]) E. LOHRMANN: *Nuovo Cimento*, **5**, 1074 (1957).

([8]) K. GOTTSTEIN and M. TEUCHER: *Zeits. f. Naturfor.*, **8a**, 120 (1953).

([9]) L. v. LINDERN: *Zeits. f. Naturfor.*, **11a**, 340 (1956).

([10]) V. D. HOPPER, S. BISWAS and J. F. DARBY: *Phys. Rev.*, **84**, 457 (1951).

([11]) A. DEBENEDETTI, C. M. GARELLI, L. TALLONE and M. VIGONE: *Nuovo Cimento*, **4**, 1142 (1956).

([12]) R. G. GLASSER, D. M. HASKIN, M. SCHEIN and J. J. LORD: *Phys. Rev.*, **99**, 1555 (1955).

([13]) P. CIOK, M. DANYSZ, J. GIERULA, A. JURAK, M. MIĘSOWICZ, J. PERNEGR, J. VRANA and W. WOLTER: *Nuovo Cimento*, **6**, 1409 (1957).

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have assumed that two tracks among them were relativistic protons, i.e., $n_{\pm} = n_s - 2$, where n_s denotes the number of shower tracks.

The data given by BARKAS *et al.* (5) are quite different in nature from the other listed in Table I. These authors have investigated extensively the antiproton-nucleon annihilation process by the nuclear emulsion technique and have found that on the average (5.3 ± 0.4) (charged and neutral) π-mesons are produced in the process. The process is equivalent in available energy to the nucleon-nucleon interaction of the Lorentz factor $\gamma_c = 2$ and the inelasticity $K = 1$ in the center-of-mass system (14), since these antiprotons came

TABLE I. – *Examples of high-energy interactions employed for the analysis.*

Shower number	Authors	Primary energy (GeV)	Number of heavily ionizing prongs, N_h	Lorentz factor in the c.m.s. γ_c	Charged meson multiplicity $n_{\pm}$	Inelasticity K	Average kinetic energy of mesons in the c.m.s. (GeV)
1	BARKAS *et al.* (5)	—	—	2.0 (a)	3.5	1.0 (a)	0.18
2	KANEKO *et al.* (6)	25	0	3.7	4	0.41	0.34
3	LOHRMANN (7)	37	0	4.5	17 (b)	1.0	0.18
4	GOTTSTEIN, TEUCHER (8)	40	0	4.7	18	1.0	0.13
5	LOHRMANN (7)	130	0	8.6	10	0.4	0.25
6	LINDERN (9)	300	0	13	8	0.28 (c)	0.37 (c)
7	LOHRMANN (5)	700	3	19	34	0.75	0.46
8	HOPPER *et al.* (10)	1000	0	24	6	0.1	0.31
9	LOHRMANN (7)	1700	0	30	10	0.31	1.0
10	DEBENEDETTI *et al.* (11)	3800	2	45	38	0.72	0.9
11	GLASSER *et al.* (12)	20000	2	100	13	0.12	1.1
12	CIOK *et al.* (13)	330000	0	420	16 (d)	0.1	a few

(*a*) The available energy in this antiproton-nucleon annihilation process is equal approximately to the mass energy of both of the particles; i.e., the process is equivalent to the case of the nucleon-nucleon collision of $\gamma_c = 2$ and $K = 1$.

(*b*) This event cannot be a pure proton-proton collision because $n_{\pm}$ is not even.

(*c*) These values have been estimated by us making use of the data of meson energies in the center-of-mass system given by Lindern.

(*d*) These authors have regarded the jet $0 + 14\alpha$ as being due to a single nucleon-nucleon collision and have estimated the number of charged particles produced in the interaction to be 12. However, taking into account an anomalously large number of π^0-mesons as compared with charged mesons they have observed, we take the value 16 as a more probable estimate for the multiplicity of charged mesons.

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(14) This equivalence may be somewhat questionable because of the large interaction cross-section observed for the former process. But the difference between them, if it exists, will be small so far as we are concerned with the process of meson production (see, for instance, N. YAJIMA and K. KOBAYAKAWA: *Progr. Theor. Phys.*, **19**, 192 (1958)).

to rest or slowed down to very small velocities before they have annihilated. The results of this experiment provide us perhaps at present with the most reliable estimate of meson multiplicity in this low energy region because of their excellent statistical accuracy and of the well-defined energy.

For the events shown in Table I we have plotted in Fig. 1 the normalized multiplicity $n_{\pm}/K$, the multiplicity of charged mesons divided by the inelasticity, as a function of $\gamma_c - 1$ which represents a half of the maximum energy available for meson production in the center-of mass system in units of the nucleon rest energy. For the same data, on the other hand, the meson multiplicity $n_{\pm}$ is plotted against the quantity $K(\gamma_c - 1)$ in Fig. 2. Of course, the latter quantity corresponds to a half of the center-of mass energy transferred to meson field in the interaction.

3. – Interpretation and discussion.

3·1. – The values of primary energy and inelasticity estimated by each author for the cosmic-ray showers of Table I should be in considerable error

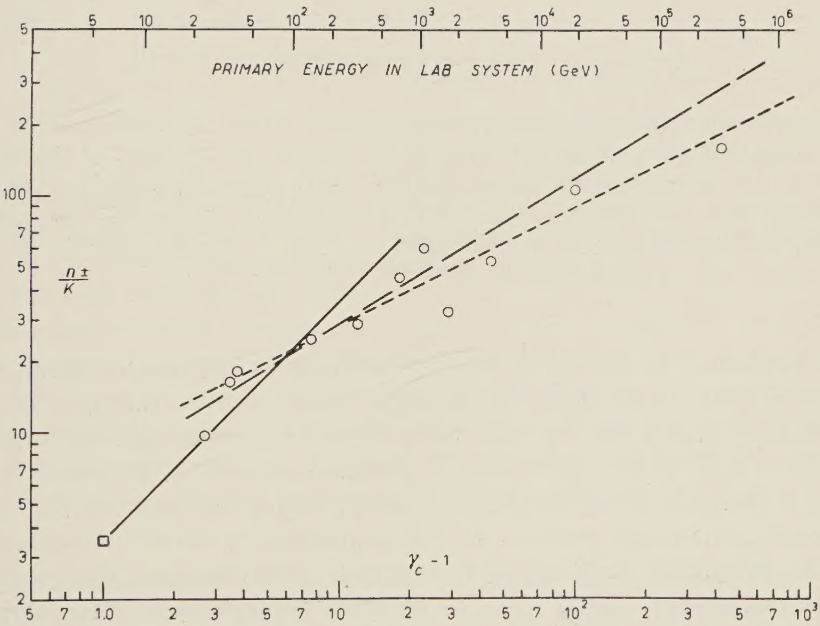

Fig. 1. – The normalized multiplicity, the multiplicity divided by the inelasticity, of charged mesons plotted against the kinetic energy (in units of the nucleon rest energy) of the incident nucleon in the center-of-mass system for the events shown in Table I. The solid curve shows the straight line of slope 1 passing through the antiproton point (marked with a square). The broken curve is that calculated by the Heisenberg theory (Eq. (7)) assuming the $\frac{5}{3}$ power spectrum of produced π-mesons. The dotted curve represents the straight line of slope $\frac{1}{2}$ fitted to the higher energy points.

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because of a severe limitation of the experimental technique and of uncertainties of the assumptions employed. Accordingly, we cannot draw the final conclusion only from a comparison between Fig. 1 and 2. It seems, however, that the experimental points shown in Fig. 1 can be fitted better to a smooth line not only than the usual plot of $n_{\pm}$ *versus* $\gamma_c - 1$ but also than the plot in Fig. 2. (A further discussion will be given later on.)

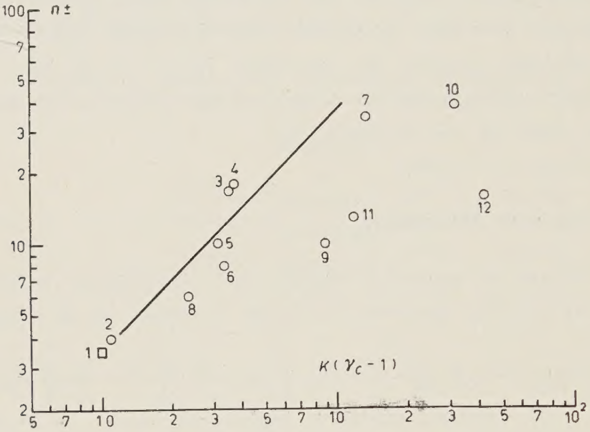

Fig. 2. – The multiplicity of charged mesons plotted against the product of the inelasticity and the kinetic energy (in units of the nucleon rest energy) of the incident nucleon in the center-of-mass system for the events shown in Table I. The figure attached to each experimental point refers to the shower number listed in column 1 of Table I. The curve is the straight line of slope 1 passing through the antiproton point (marked with a square).

In particular, as shown in Fig. 1, the straight line passing through the antiproton point with slope 45° is fitted fairly well to the data at lower energies (less than a few hundred GeV). This line results evidently from the assumption that the average energy of produced mesons in the center-of-mass system is independent of the primary energy and is the same as that in the antiproton annihilation process. In this connection, it should be remembered that the antiproton data are in fact reliable. Furthermore, the cosmic-ray data in this energy region seem to be more accurate than those at higher energies, since the former events can perhaps definitely be regarded as due to pure proton-proton collisions except for shower 3 and also the energy measurements can be done with more ease in general at lower energies. Therefore, it seems reasonable to conclude, at least within the present experimental limitation, that the average energy of produced mesons is nearly constant independently of the primary energy at least up to about 100 GeV so that the

meson multiplicity shows a rather sharp increase with the primary energy; i.e., we obtain the empirical relation

$$n_{\pm} \simeq 3.5\,K(\gamma_c - 1)\,, \qquad \text{for } \gamma_c \lesssim 7 \tag{1}$$

It is also shown that the experimental information obtained simply from the angular distribution of shower particles seems to support the above expression. If we assume that the π-mesons are emitted with a unique energy $\bar{\gamma}_\pi$ (less than γ_c) in the center-of-mass system, the maximum angle of shower particles in the laboratory system, θ_m, is given approximately by [15]

$$\sin\theta_m = \left(\frac{\bar{\gamma}_\pi^2 - 1}{\gamma_c^2 - 1}\right)^{\frac{1}{2}} \simeq \frac{\bar{\gamma}_\pi}{\gamma_c}\,.$$

From a simple energetical consideration, we then obtain

$$\frac{n_\pm}{K} \simeq \frac{9}{\sin\theta_m} \cdot \frac{\gamma_c - 1}{\gamma_c}\,. \tag{2}$$

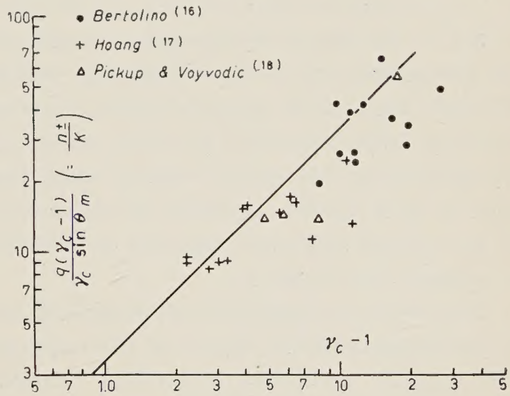

Fig. 3. – The quantity $9(\gamma_c - 1)/(\gamma_c \sin\theta_m)$ measured simply from the secondary angular distribution is plotted as a function of $\gamma_c - 1$ for cosmic-ray showers observed by various authors. The straight line is the same as the solid curve in Fig. 1 (Eq. (1)).

The right hand side of this equation can be determined from the data of secondary angular distribution by the usual method. (The median angle method gives a systematic error for estimating the value of γ_c in the case of $\gamma_c > \bar{\gamma}_\pi$, but the error is very small [15] for $\bar{\gamma}_\pi \gtrsim 2$.) In Fig. 3, we have plotted the experimental value of this quantity against $\gamma_c - 1$ for the showers observed by BERTOLINO [16], HOANG [17], and PICKUP and VOYVODIC [18], which involved showers initiated by nucleon-nucleus collisions ($N_h \lesssim 4$). The fact

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[15] C. C. DILWORTH, S. J. GOLDSACK, T. F. HOANG and L. SCARSI: *Nuovo Cimento*, **10**, 1261 (1953).

[16] G. BERTOLINO: *Nuovo Cimento*, **3**, 141 (1956). The showers with large inelasticities (belonging to zone A) observed by him have not been plotted here. Recent analysis made by I. A. IVANOVSKAYA and D. S. CHERNAVSKY (*Nucl. Phys.*, **4**, 29 (1957)) indicates that these showers were produced presumably by the collision of an incident nucleon with two or more target nucleons.

[17] T. F. HOANG: *Journ. Phys. Rad.*, **14**, 395 (1953).

[18] E. PICKUP and L. VOYVODIC: *Phys. Rev.*, **84**, 1190 (1951).

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that the straight line of Eq. (1) fits fairly well to most of the lower energy points out of them implies

$$\bar{\gamma}_\pi = \gamma_c \sin\theta_m = \text{const } (\simeq 2.4)\,,$$

as can be seen from Eq. (2). It should be remarked that these data which were obtained simply from the secondary angular distribution are in fair agreement with those which were analysed in more detail under the stringent selection criteria ([19]).

3·2. – At higher energies, deviations of the experimental points from the 45° line can clearly be seen from Fig. 1 (Fig. 3 gives also a similar indication). This fact leads us to suggest that the increase of the average energy of mesons resulting from increasing primary energy takes place at about 100 GeV. The dependence of the average energy upon the primary energy suggested by this view is as a matter of course in substantial agreement with a general trend of the values estimated directly by the respective authors which are shown in column 8 of Table I.

This is also consistent with experimental results on the angular distribution of produced mesons. Many of the experiments so far performed by means of photographic emulsions and cloud chambers show a nearly isotropic distribution up to about 100 GeV primary energy ([1,8,9,20]) and some degree of anisotropy at greater energies ([21]). At extremely high-energies, a strong anisotropy has been well established ([11-13]). It seems, therefore, that the increase of the average energy is closely connected with the appearance of the angular anisotropy.

This view would also be reasonable from theoretical considerations. A possible strong interaction of produced π-mesons with each other may result in a fairly constant value of the average energy of a number of mesons with an isotropic angular distribution at lower primary energies, whereas at higher energies ($\gamma_c \gtrsim 7$) the Lorentz contraction of the interaction volume may play an essential role resulting in a collimation of high-energy mesons into the

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([19]) By analising his own data, BERTOLINO ([16]) has obtained the empirical relation

$$n_s - 1 = 3.33K(\gamma_c - 1) + 0.975\,,$$

which is quite similar to Eq. (1).

([20]) For example, R. GIACCONI, A. LOVATI, A. MURA and C. SUCCI: *Nuovo Cimento*, **4**, 826 (1956); Y. WATASE, S. MITANI, K. SUGA and Y. TANAKA: *Progr. Theor. Phys.*, **18**, 314 (1957).

([21]) S. TOKUNAGA: *Journ. Phys. Soc. Jap.*, **12**, 454 (1957); W. WINKLER: *Helv. Phys. Acta*, **29**, 267 (1956).

forward and backward cones, as was explained by HEISENBERG [22,23] or LANDAU [24], in addition to a nearly isotropic ejection of lower energy mesons. In fact, this expectation is consistent with many experimental evidence [7-12] that a large fraction of secondary particles have been emitted with relatively low energies even in energetic collisions.

3·3. – In order to compare the present results with the theories of multiple meson production, let us confine ourselves only to the data at higher energies. It appears that the straight line of slope $\frac{1}{2}$ is fitted fairly well to showers 5 to 12 as shown by the dotted curve in Fig. 1; i.e., we have empirically

$$n_{\pm} \simeq C\,K(\gamma_c - 1)^m\,, \qquad \text{for } \gamma_c \gtrsim 7, \tag{3}$$

with the tentative values of

$$\begin{cases} C \simeq 9 \\ m \simeq 0.5\,. \end{cases} \tag{3a}$$

On the other hand, any straight line in Fig. 2 is written

$$n_{\pm} = C'[K(\gamma_c - 1)]^{m'}\,, \tag{4}$$

where C' and m' are constants. If we assume that only π-mesons are produced in the interaction, it is evident that Eq. (3) holds in the case when the average value of the total energies of the π-mesons in the center-of-mass system is given by

$$\bar{\gamma}_\pi \propto (\gamma_c - 1)^{1-m}\,, \tag{5}$$

i.e., when it depends primarily on the Lorentz factor of the colliding system. On the other hand, if the average energy depends on the energy transferred to the meson field in such a manner as

$$\bar{\gamma}_\pi \propto [K(\gamma_c - 1)]^{1-m'}\,, \tag{6}$$

then the multiplicity is given by Eq. (4). It is possible that the main difference between Eqs. (3) and (4) can be given by the physical contents of Eqs. (5) and (6), even if other particles (heavy mesons and nucleon pairs)

(22) W. HEISENBERG: *Zeits. Phys.*, **126**, 569 (1949).
(23) W. HEISENBERG: *Zeits. Phys.*, **133**, 65 (1952); *Kosmische Strahlung* (Berlin, 1953), p. 148.
(24) S. Z. BELEN'KJI and L. D. LANDAU: *Suppl. Nuovo Cimento*, **1**, 15 (1956).

having different average energies from that of π-mesons come to play a role, since a contribution of these particles to the multiplicity is likely to be fairly small even at extremely high-energies [25].

As already mentioned, the present data do not permit us to judge definitely this alternative, nevertheless we intend to give a preference to Eq. (5) by the following reasons. First, from Fig. 2 an inference could be drawn that the higher energy points (say, showers 9 to 12) are considerably below the straight line of slope 45° which is well fitted to the lower energy points (showers 1 to 5), so that a smooth line capable to be fitted to all the experimental points in the whole energy region could never be drawn. Furthermore, the assumption implied by Eq. (6) that the average energy is determined by the transferred energy alone rather than the Lorentz contraction of interacting fields would be less plausible on theoretical grounds. Lastly, an other support for this preference might be found in the fact that several examples of meson showers of very high multiplicity have been observed thus far at relatively low energies [4,26], since these events might otherwise not be explained even if we assumed the occurrence of cascade processes inside the target nucleus. It seems probable, therefore, that in the higher energy region the average energy of mesons in the center-of-mass system is determined primarily by the Lorentz factor of the colliding particles so that the energy dependence of multiplicity is given approximately by Eq. (3).

4. – Comparison with the theories of Heisenberg and Landau.

On the point of view described above, we have to call attention to the Heisenberg theory of multiple meson production [22,23]. This is because in his theory the maximum and hence the average energies of a particular kind of mesons are determined by the Lorentz factor of the incident particle and also because the inelasticity is introduced explicitly into the formulation. The theory can be characterized by the power law energy spectrum of mesons produced:

$$\mathrm{d}n = \mathrm{d}\gamma_\pi/\gamma_\pi^{1+a}\,.$$

If we assume only the production of π-mesons with $\alpha = \frac{2}{3}$, for example, the multiplicity of charged mesons can be written approximately [22]

$$n_\pm = 4.5K(\gamma_c - 1)(1 - \gamma_c^{-\frac{2}{3}})(\gamma_c^{\frac{1}{3}} - 1)^{-1}\,. \tag{7}$$

([25]) F. A. Brisbout, C. Dahanayake, A. Engler, Y. Fujimoto and D. H. Perkins: *Phil. Mag.*, **1**, 605 (1956).

([26]) M. Schein, D. M. Haskin and R. G. Glasser: *Suppl. Nuovo Cimento*, **12**, 355 (1954).

This equation is shown by the broken line in Fig. 1, which is not inconsistent with the experimental results. We may obtain a better fit by taking into account the production of heavy mesons and nucleon pairs. However, the inverse square law ($\alpha = 1$) of the energy spectrum [23] seems to give too high a multiplicity at the highest energy.

A comparison between the present results and the Landau theory [24] can be made under the assumption that the completely inelastic ($K = 1$) interaction corresponds to the case of a head-on collision in the theory. The theory is apparently in good agreement with the fact that the empirical value of the exponent m is approximately $\frac{1}{2}$ (Eqs. (3) and (3a)), but the numerical coefficient $C \simeq 1$ predicted by the theory is too small in comparison with the empirical value $C \simeq 9$. However, by introducing the effective inelasticity as evaluated by Amai *et al.* [27] into the theory, we may obtain a better agreement with the experiments. Therefore, the present analysis could not permit us to discriminate between the theories of Heisenberg and of Landau.

5. – Conclusion.

The results obtained above are summarized as follows:

i) By making use of the normalized multiplicity of mesons produced, we may explain consistently various high-energy showers in terms of the single nucleon-nucleon interactions.

ii) At lower primary energies (from 10 to about 100 GeV), it seems that the average energy of the produced mesons in the center-of-mass system is approximately the same as that in the antiproton-nucleon annihilation process (about 200 MeV kinetic energy) independently of the primary energy and the inelasticity. Thus, the normalized multiplicity increases sharply with the primary energy (Eq. (1)).

iii) At energies greater than 100 GeV, say, the average meson energy seems to increase with the primary energy resulting in a slower increase of multiplicity. These circumstances have, at least to some extent, relation to the experimental evidence that the angular distribution of mesons is nearly isotropic up to about 100 GeV primary energy and it also shows a strong anisotropy at extremely high-energies.

iv) Though any decisive conclusions should be reserved because of experimental uncertainties, the energy dependence of multiplicity at higher

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(27) S. Amai, H. Fukuda, C. Iso and M. Sato: *Progr. Theor. Phys.*, **17**, 241 (1957).

energies may be approximated empirically by Eqs. (3) and (3*a*). The Heisenberg and the Landau theories, in a somewhat modified form, are shown to be not inconsistent with the present results.

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We wish to express our sincere thanks to Professor Y. WATASE for his kind interest and encouragement. We are also indebted to Prof. S. HAYAKAWA, Prof. K. SUGA, Mr. O. KUSUMOTO and Mr. S. MATSUMOTO for their stimulating discussion and criticism.

RIASSUNTO (*)

Si analizzano alcuni dati sperimentali su sciami di alta energia finora resi noti. Si dimostra che, tenuto conto delle fluttuazioni dell'anelasticità, appare possibile spiegare fenomenologicamente la dipendenza della molteplicità mesonica dall'energia primaria, almeno nei limiti dell'esattezza attuale della sperimentazione, in termini della singola collisione nucleone-nucleone. Il risultato sembra accordarsi con l'ipotesi che l'energia media dei mesoni prodotti è abbastanza costante (energia cinetica circa 200 MeV) indipendentemente dall'energia primaria, almeno fino a circa 100 GeV (nel sistema del laboratorio), cosicchè la molteplicità mesonica normalizzata al valore 1 dell'anelasticità cresce piuttosto bruscamente con l'energia primaria. Ad energie superiori, la collimazione dei mesoni di alta energia nei coni anteriore e posteriore risulterebbe nell'accrescere l'energia mesonica media, cosicchè l'aumento della molteplicità normalizzata con l'energia primaria sembra esser più lento che alle basse energie. Si confrontano i risultati dell'analisi con quelli delle attuali teorie della produzione multipla dei mesoni.

(*) *Traduzione a cura della Redazione.*

Photons, Gravitons and the Cosmological Constant.

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(ricevuto il 28 Novembre 1957)

Summary. — An attempt is made to solve Maxwell equations in a Riemannian space subjected to $R_{ik} = \Lambda g_{ik}$: It is shown that the equation of motion for an electromagnetic wave-packet is similar to that of a particle (photon) having rest mass $(\hbar\sqrt{\Lambda}/c)$. We then deal with the equation $R_{ik} = \Lambda g_{ik}$ itself, and seek for a solution having the form of gravitational waves. It is found that, if such waves can carry quanta of energy, then the rest mass of an elementary wave packet (a graviton) is $(\hbar\sqrt{-2\Lambda}/c)$. It may therefore be concluded that Λ cannot be negative, and if gravitons do exist, it can also not be positive. It is therefore necessarily zero, and the rest masses of photons and gravitons are also null.

1. – Electromagnetic waves in curved space [1].

We seek a solution of Maxwell equations in vacuo:

$$F^{ik}{}_{;k} = 0$$

$$F^{ik} = g^{im} g^{kn} F_{mn}$$

$$F_{mn} = A_{m;n} - A_{n;m}$$

the space-time metrics being subjected to

$$R_{ik} = \Lambda g_{ik} ,$$

(1) A less rigorous derivation of the same result has been given by the author in *Compt. Rend.*, **239**, 1023 (1954).

i.e. in the approximation of weak electromagnetic fields. We have

$$F^{ik}_{\ ;k} = (A^{i;k} - A^{k;i})_{;k} = 0$$

or, lowering index i:

$$A_{i;k}^{\ \ :k} = A^{k}_{\ ;i;k} .$$

But (²)

$$A^{k}_{\ ;i;k} = g^{kh} A_{h;i;k} = g^{kh}(A_{h;k;i} + A_s R^{s}_{hik}) = A^{k}_{\ ;k;i} - A_s R^{s}_{i} .$$

In the last expression, the first term is null, because of Lorentz gauge condition $A^{k}_{\ ;k} = 0$, and the second is

$$A_s R^s_i = \Lambda A_s \delta^s_i = \Lambda A_i .$$

Therefore

$$A_{i;k}^{\ \ :k} = -\Lambda A_i .$$

This vector equation is very difficult to solve directly, because we cannot set a system of cartesian co-ordinates, and derivatives of the Γ^m_{np} field will appear on the left hand member. Moreover, in curved co-ordinates, as the vector A_i moves away, it undergoes an apparent change of direction which prevents us from defining a velocity for its components. As a matter of fact, the apparent velocity can be different for the various component of A_i, and it can also be modified by infinitesimal changes of the co-ordinates.

It is therefore better to seek for the equation of motion of some characteristic scalar tied to A_i: The simplest such scalar is of course the magnitude A of A_i:

$$A = \sqrt{g^{ij} A_i A_j} ,$$

We have

$$A_{;k} = \frac{g^{ij} A_i A_{j;k}}{A} ,$$

$$A_{;k}^{\ \ ;k} = \frac{g^{ij} A_i A_{j;k}^{\ \ ;k}}{A} + g^{ij} A_{j;k} \left(\frac{A_i}{A}\right)^{;k} .$$

(²) Much care is to be taken of the signs of the curvature tensor components, as they vary according to different authors. In the present paper, we shall follow TOLMAN *Relativity, Thermodynamics and Cosmology* (Oxford, 1934) and define

$$R^i_{jkh} = \Gamma^i_{hj,k} - \Gamma^i_{jk,h} + \Gamma^s_{jh}\,\Gamma^i_{sk} - \Gamma^s_{jk}\,\Gamma^i_{sh}$$

and

$$R_{jk} = R^i_{jki}$$

But

$$g^{ij}A_iA_{j;k}{}^{;k} = -\Lambda g^{ij}A_iA_j = -\Lambda A^2\,,$$

thus

$$A_{;k}{}^{;k} + \Lambda A = g^{ij}A_{j;k}\left(\frac{A_i}{A}\right)^{;k}.$$

We can now choose, at the point which we are considering, a system of almost cartesian co-ordinates, where the Γ^m_{np} ocally vanish; they will also be vanishingly small in its meighborhood: as a matter of fact, if we consider a wave packet the dimensions of which need not exceed a few wavelengths, the Γ terms will have order of magnitude $\lambda\Lambda$ in the region of interest.

Let us then seek a solution $A_i = B_i f$, where the B_i are slowly varying functions (3) and f is a scalar (presumably satisfying some wave-like equation). We have

$$A_{j;k} = B_j f_{,k} + B_{j;k} f = B_j f_{,k} + \frac{A}{B}\cdot B_{j;k}\,,$$

$$\left(\frac{A_i}{A}\right)_{;h} = \left(\frac{B_i}{B}\right)_{;h} = \frac{B_{i;h}}{B} - \frac{B_i g^{mn}B_m B_{n;h}}{B^3}\,,$$

where $B = \sqrt{g^{mn}B_mB_n}$. It is easily seen that

$$g^{ij}B_j\left(\frac{B_i}{B}\right)^{;h} = 0\,,$$

so that

$$A_{;k}{}^{;k} + \Lambda A = Ag^{ij}g^{kh}\frac{B_{j;k}}{B}\left(\frac{B_i}{B}\right)_{;h}.$$

Thus, if we can find slowly varying B_i (3) the right hand side of the last equation will have order of magnitude $A\lambda^2\Lambda^2$, i.e. will be vanishingly small even compared with the ΛA term in the left-hand side. That such B_i can be found may be shown by developing the vector wave equation $A_{i;k}{}^{;k} + \Lambda A_i = 0$, in which we replace A_i by $A(B_i/B)$. We then have

$$A_{;k}{}^{;k}\left(\frac{B_i}{B}\right) + 2A_{;k}\left(\frac{B_i}{B}\right)^{;k} + A\left(\frac{B_i}{B}\right)_{;k}{}^{;k} + \Lambda A\left(\frac{B_i}{B}\right) = 0\,,$$

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(3) I.e. in $B_{i;j} = B_{i,j} - B_s\Gamma^s_{ij}$ both terms on the right-hand side are of the same order of magnitude $B\lambda\Lambda$.

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or, with the help of the just derived value of $A_{;k}^{\ ;k}$

$$\left(\frac{B_i}{B}\right) g^{lj} \frac{B_{j;k}}{B} \left(\frac{B_l}{B}\right)^{;k} + 2\frac{A_{.k}}{A}\left(\frac{B_i}{B}\right)^{;k} + \left(\frac{B_i}{B}\right)_{;k}^{\ ;k} = 0$$

Each term of this equation involves covariant derivatives of B_i, so that none is of zero order in $\lambda\Lambda$ if B_i indeed has the form $B_i = C_i + D_i$, where the C_i are arbitrary constants and the D_i are small terms of order $C_i\lambda^2\Lambda$. There remain only first (and higher) order terms, and the four equations can be solved for the four D_i in a way consistent with the hypothesis that they are small.

Once this point has been settled, we can go on with the scalar wave-equation

$$A_{;k}^{\ ;k} + \Lambda A = 0 .$$

This equation is known to represent waves travelling with group-velocity

$$v = c\sqrt{1 - \frac{\Lambda c^2}{4\pi^2\nu^2}} ,$$

where ν is the frequency.

And if we have such a wave packet containing only one photon of energy $W = h\nu$ we can consider its rest mass as being

$$\mu_p = \frac{W}{c^2}\sqrt{1-\frac{v^2}{c^2}} = \frac{h\sqrt{\Lambda}}{2\pi c} .$$

2. – Gravitational waves in curved space.

We shall now deal with the equation $R_{ik} = \Lambda g_{ik}$ itself. We have

$$R_{ik} = \frac{1}{2} g^{mn} \frac{\partial^2 g_{ik}}{\partial x^m \partial x^n} - \frac{1}{2}\frac{\partial}{\partial x^i}(g^{mn}\Gamma_{kmn}) - \frac{1}{2}\frac{\partial}{\partial x^k}(g^{mn}\Gamma_{imn}) + \text{terms quadratic in } \Gamma .$$

As previously, we choose geodetic co-ordinates, such that the Γ vanish at the point which we are considering, and have magnitude $\lambda\Lambda$ in a neighborhood of dimensions λ. We can therefore neglect terms quadratic in Γ.

Moreover, we can impose on the field four additional conditions (4) such as

$$g^{mn}\Gamma_{lmn} = 0 .$$

(4) Cfr. EINSTEIN: *The Meaning of Relativity* (Princeton, 1953), p. 87.

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(Note that these conditions are to hold exactly, and not only approximately, in finite region of space-time. This is quite similar to the use of Lorentz gauge condition $A^k{}_{;k}=0$ in the previous section). It then remains

$$\frac{1}{2}\, g^{mn}\, \frac{\partial^2 g_{ik}}{\partial x^m\, \partial x^n} = \Lambda g_{ik}\, .$$

As usual, we write $g_{ik}=\gamma_{ik}+h_{ik}$, where $\gamma_{ik}=\pm 1,\ 0$ are the galilean values for g_{ik} and h_{ik} are small quantities, the square of which can be neglected. Then

$$\frac{1}{2}\, \gamma^{mn}\, \frac{\partial^2 h_{ik}}{\partial x^m\, \partial x^n} = \Lambda(\gamma_{ik} + h_{ik})\, .$$

Let us now see how a perturbation δh_{ik} of the h_{ik} propagates. (This perturbation has to be small, of course, relative to the γ_{ik} but not necessarily with respect to the h_{ik}).

We have

$$\frac{1}{2}\, \gamma^{mn}\, \frac{\partial^2 (h_{ik} + \delta h_{ik})}{\partial x^n\, \partial x^n} = \Lambda(\gamma_{ik} + h_{ik} + \delta h_{ik})\, .$$

We can substract from this the equation for the unperturbed h_{ik} (which represents the « natural » curvature of the field) and get

$$\frac{1}{2}\, \gamma^{mn}\, \frac{\partial^2 (\delta h_{ik})}{\partial x^m\, \partial x^n} = \Lambda\ (\delta h_{ik})\, .$$

Once more we get a Klein-Gordon-like equation, which corresponds to waves travelling with group velocity

$$v = c\sqrt{1+\frac{2\Lambda c^2}{4\pi^2 \nu^2}}\, ,$$

and if there do exist wuch wave-packets carrying quanta of energy (gravitons) $W=h\nu$, their rest mass will be ([5])

$$\mu_g = \frac{W}{c^2}\sqrt{1-\frac{v^2}{c^2}} = \frac{h\sqrt{-2\Lambda}}{2\pi c}\, .$$

([5]) This result had already been found, in a less direct way, by TONNELAT (cfr. DE BROGLIE: *Théorie générale des particules à spin* (Paris, 1954), p. 191).

Much care should be taken of signs: our R_{ik} is equal to the R_{ik} there, but our g_{ik} is minus the g_{ik} there. Therefore the definitions for the Λ's are opposite in signs, which explains the difference in the final formulae.

3. – Conclusions.

As it is a fact of experience that electromagnetic waves can carry quanta of energy, we may conclude that Λ cannot be negative (unless we bring some modifications to Maxwell or Einstein's equations). If it is not null, its value is less than $5 \cdot 10^{-54}$ cm^{-2} [6], so that the rest mass of the photon is certainly less than 10^{-64} g.

On the other hand, it has still not been proved that gravitational waves can carry quanta of energy. If they can, then Λ cannot also be positive. It is therefore necessarily zero, and the rest masses of photons and gravitons are also null.

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The author is indebted to Prof. N. Rosen for valuable suggestions and clarifying discussions of the results.

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(6) Cfr. Tolman: *Relativity, Thermodynamics and Cosmology* (Oxford, 1934), p. 474.

RIASSUNTO (*)

Cerchiamo una soluzione delle equazioni di Maxwell in uno spazio Riemanniano per il caso $R_{ik} = \Lambda g_{ik}$. Dimostriamo che l'equazione del moto di un pacchetto d'onde elettromagnetiche è simile a quella di una particella (fotone) con massa a riposo $\hbar\sqrt{\Lambda}/c$. Trattiamo poi l'equazione $R_{ik} = \Lambda g_{ik}$ e cerchiamo una soluzione in forma d'onde gravitazionali. Si trova che, se tali onde possono trasportare quanti di energia, la massa a riposo di un pacchetto d'onde elementare (gravitone) è $\hbar\sqrt{-2\Lambda}/c$. Si può pertanto concludere che Λ non può essere negativo e che, se i gravitoni esistono, non può neanche essere positivo. È pertanto necessariamente zero e le masse a riposo dei fotoni e dei gravitoni sono a loro volta nulle.

(*) *Traduzione a cura della Redazione.*

Cosmic-Ray Evidence on the Origin of Meteorites (*).

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(ricevuto il 19 Gennaio 1958)

Summary. — The birth of a meteorite coincides with the breakup of its parent body and also with the beginning of the meteorite's exposure to cosmic rays. This continued exposure creates through nuclear evaporations definitely calculable amounts of ^{3}He, about $\frac{2}{3}$ of it by way of tritium. Hence the date of breakup can be obtained from the ^{3}He content; or better still, from the ^{3}He and tritium content measured in the same sample. However, the latter method has only limited applicability. The two methods give good accord; of six meteorites dated, all are found to be about 300 million years old. This agreement establishes confidence in the calculations of the ^{3}He production rate and leads to some considerations concerning the process of nuclear excitation. The age is also in good accord with Öpik's calculation of the mean lifetime of meteorite-like bodies against capture by the inner planets. From this agreement conclusions can be drawn concerning the mechanism of meteorite creation and concerning the relative constancy of cosmic-ray intensity as far back as 300 million years.

1. – Introduction.

It seems fairly well established, especially from the metallurgical evidence on iron meteorites, that meteorites were formed by slow cooling and under high pressure [1,2]. Suitable conditions prevail only in the interior of good-sized bodies (planetoids) which may have existed at one time between the orbits of Mars and Jupiter in the present asteroid belt. According to current

(*) Presented as an invited paper at the 1957 Spring meeting of the American Physical Society; supported in part by the Office of Scientific Research, U. S. Air Force.

(1) H. H. UHLIG: *Geochim. et Cosmochim. Acta*, **6**, 228 (1954).

(2) E. J. ÖPIK: *Irish Astron. Journ.*, **3**, 206 (1955).

theories, mainly associated with the names of Kuiper and Urey, these planetoids were formed at the beginning of the solar system, first as dust aggregates; they condensed under gravitational attraction, were heated by radioactivity, and liquified. Within a very short time, by geological standards, these small bodies cooled. In this process the stony components rose to the top while the iron-nickel mass remained at the center. This differentiation, which is also thought to exist in the earth, gives rise to the two main types of meteorites, stones and irons. In the slow cooling of the core, the characteristic crystal structures of iron meteorites were formed showing the so-called Widmannstätten figures. Some time after, these planetoids were broken up by collisions and produced what we now call the asteroids. The term « meteorites » is applied to those asteroids which have fallen on the earth or on the moon. Experimental evidence, both from the lead isotope method ([3]) and from the argon-potassium method ([4]) of radioactive dating, is now accumulating, showing that the *solidification* of the planetoids took place around $4 \cdot 19^{9}$ years ago; this corresponds to about the age of the solar system.

A separate problem and one to which the present paper is devoted is the date of *breakup* of the planetoids, i.e., the date of creation of the meteorites. The method we use depends on the fact that up until breakup the (potential) meteorites are in the interior of the planetoid and are shielded from cosmic rays; but after breakup they become exposed to cosmic rays. The cosmic-ray exposure time thus gives us a method of dating the creation of a meteorite ([5]).

2. – Methods of dating.

2·1. *Helium-3 Dating.* – After it was realized that cosmic-ray bombardment of meteorites would produce substantial and measurable amounts of ^{3}He, the possibility of using a meteorite as an integrating cosmic-ray meter became immediately apparent ([6]). Dividing the calculated production rate of ^{3}He into the measured amount of ^{3}He, one immediately obtains the time interval during which the meteorite has been exposed to the cosmic-ray flux (assumed to be of constant intensity), and therefore the date of creation of the meteorite from the breakup of its parent planet. This approach has been used to give

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([3]) C. Patterson, H. Brown, G. Tilton and M. Inghram: *Phys. Rev.*, **92**, 1234 (1953).

([4]) E. K. Gerling and T. G. Pavlova: *Dokl. Akad. Nauk USSR*, **77**, 85 (1951); G. J. Wasserburg and R. J. Hayden: *Phys. Rev.*, **97**, 86 (1955); S. J. Thomson and K. I. Mayne: *Geochim. et Cosmochim. Acta*, **7**, 169 (1955); W. Gentner and W. Kley: *Zeits. Naturfor.*, **10***a*, 832 (1955).

([5]) S. F. Singer: *Astrophys. Journ.*, **119**, 291 (1954).

([6]) S. F. Singer: *Nature*, **170**, 728 (1952).

so-called « cosmic-ray ages » for a number of meteorites ([7]), or at least a lower limit. It was found for example, that meteorites with the highest ^{3}He content had breakup ages of about 300 million years, or possibly more (Table I).

TABLE I. – *Compilation of age determinations for various meteorites.*

	Total He content (10^{-6} cm^3/g)	Preatmospheric diameter [a] (cm)	^{3}He content (10^{-6} cm^3/g)	Estimated « geometric factor »	Duration of cosmic-ray exposure (yr)
Mount Ayliff	36.8	14.6	8.8 [a]	$2 \div 3$	$200 \div 300 \cdot 10^6$
Tamarugal	23.6	42	5.57 [a]	$\lesssim 1$	$\gtrsim 340 \cdot 10^6$
Carbo	22.0	48	4.5 [a]	~ 1	$\sim 300 \cdot 10^6$
Toluca (Durham)	18.9	Very large	4.33 [a]	$\lesssim 1$	$\gtrsim 290 \cdot 10^6$
Thunda	29.0	Small but irregular	6.8 [b]	$1 \div 2$	$200 \div 400 \cdot 10^6$
Norton County (Stone) } same fall			2.27 ± 0.11 [c]		$240 \cdot 10^6$ [c]
Furnas County (Stone) } same fall			2.35 ± 0.11 [c]		$280 \cdot 10^6$ [c]

(*a*) Reference ([8]).
(*b*) P. REASBECK and K. I. MAYNE: *Nature*, **176**, 733 (1955).
(*c*) Reference ([17]).

But to maintain this view, it had first to be shown that the low *solidification* ages (of about 100-200 million years) derived from the ^{4}He-uranium method ([8]) were invalid, either (i) because of the absence of uranium ([9,10]) or (ii) because of the leakage of radiogenic ^{4}He ([7,11,12]).

([7]) S. F. SINGER: *Sci. Amer.*, **191**, 36 (1954).
([8]) J. DALTON, F. A. PANETH, P. REASBECK, S. J. THOMSON and K. I. MAYNE: *Nature*, **172**, 1168 (1953).
([9]) H. C. UREY: *Nature*, **175**, 321 (1955).
([10]) G. W. REED and A. TURKEVICH: *Nature*, **176**, 794 (1955).
([11]) S. F. SINGER: *Phys. Rev.*, **105**, 765 (1957).
([12]) A crucial experiment ([11]) has been suggested to test hypothesis (ii). If it turns out to be correct, then we suggest the following interpretation of the U-^{4}He data: the meteorite fragments which are formed in the break-up are heated throughout by the shock wave of the collision. In many meteorites this heating is not high enough to destroy the crystal structure; but it is sufficient to accelerate the diffusion of the radiogenic helium which is located in the boundaries of the crystal grains, and thereby to give an *apparent* recent solidification age. This suggestion appears to us less *ad hoc* than the one put forward by P. REASBECK and K. I. MAYNE (*Nature*, **176**, 186 (1955)), namely that the iron phase solidified only about 200 million years ago, and that the breakup occurred soon after this solidification.

The calculation of the ^{3}He production rate in a particular meteorite, and therefore the derivation of its breakup time, is made difficult by the fact that we do not know the shape and size of the meteorite before it enters the atmosphere, nor do we generally know the position of the sample with respect to the preatmospheric boundary. In most cases, therefore (e.g., for the meteorites Tamarugal and Toluca in Table I), we can only give a lower limit to the age. However, for the meteorite Carbo, helium determinations have been made as a function of depth [13], and for this meteorite we can estimate the geometric factor [14], and therefore the age, quite well.

In many cases we can only say that the sample came from the interior of the meteorite and therefore experiences a production rate less than that at the maximum, i.e., we estimate a geometric factor of less than one. For very small meteorites, however, the cosmic-ray flux could have penetrated over a larger solid angle; here we must estimate a geometric factor greater than one. For example, for the meteorite Mt. Ayliff we estimate a geometric factor of the order of 2 to 3. The uncertainties, while large, do not change our conclusions substantially.

The uncertainties arise as follows:

1) It is very probable that Mt. Ayliff did not enter as a small spherical meteorite as for example Paneth [8] calculated. It may have formed part of the skin of a larger meteorite from which it was spalled off. In that case the geometric factor would be only one. This seems to have happened in the case of the Toluca meteorite: the sample at Durham has a very high ^{3}He content while the sample in Hamburg has a ^{3}He content less than 1% of Durham [8], clearly indicating that the Durham sample came from the region of the skin whereas the Hamburg sample came from the interior. We therefore feel justified in assuming that the geometric factor for Toluca (Durham) is not very much less than one, and that the age cannot be very much greater than 300 million years.

2) But there exist in addition a large number of meteorites whose ^{3}He content is very small, in some cases unmeasurable. We cannot at the present time tell whether these form the interior portions of larger meteorites, i.e. a geometric factor much less than one, or whether they are meteorites which were created by breakup only a relatively short time ago. In other words, we do not know very much yet concerning the dispersion of ages and the results of Table I should not be used to indicate a unique age for all meteorites.

3) A further uncertainty of course, is the accuracy of the calculated value [6] for the ^{3}He production rate. Other investigators have derived much lower rates than ours and find therefore correspondingly higher breakup ages. For Mt. Ayliff Fireman [15]

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(13) F. A. Paneth, P. Reasbeck and K. I. Mayne: *Nature*, **172**, 200 (1953).

(14) Defined as the ratio of ^{3}He production rate in the sample to the rate at the transition maximum, i.e., $(2 \div 15)$ cm from the preatmospheric boundary of a very large iron meteorite.

(15) E. L. Fireman: *Phys. Rev.*, **97**, 1303 (1955).

quotes an age of 4000 million years while CURRIE, LIBBY and WOLFGANG [16] give its age as 1400 million years. In both cases the ^{3}He production rate was calculated from experiments on the production of tritium in cyclotron-irradiated targets. (Possible reasons for the discrepancy are discussed below).

2·2. *Tritium-Helium-3 Dating.* – In a recent experiment BEGEMANN, GEISS and HESS [17] have applied the following method: in recently fallen stone meteorites (Norton County and Furnas County) where a measurable tritium activity existed, both the tritium and ^{3}He contents were measured. From the tritium content they can calculate the production rate for tritium, and, by derivation, for ^{3}He (Tables II and III). These measured production rates are in good accord with our calculations, as discussed below. Again under the assumption of constant cosmic-ray intensity, they derive a cosmic-ray age of about 260 million years, a value which is in general accord with our age determinations [7] (see Table I).

It is evident that the tritium-^{3}He method overcomes the major disadvantage of the ^{3}He method, namely the uncertainty of the position of the sample relative to the preatmospheric boundary. On the other hand, there are very few cases of recent falls to which the tritium-^{3}He method can be applied. It seems to us, therefore, that the determination of ^{3}He as a function of depth [13] can be used most widely as a means of «locating» the sample and refining the age estimates. Ideally, the depth-variation method should be checked by means of the tritium method and a consistent calibration obtained.

3. – Discussion.

The good accord between the age deduced from the ^{3}He-tritium method and from our ^{3}He calculations is of course very gratifying. One needs to explain, however, (1) why the direct measurements [15,16] of tritium production in a cyclotron are so much lower, and (2) how one one can explain the fact that the meteorite ages are as low as 300 million years. (3) Finally, we wish to make some remarks concerning the prehistoric cosmic-ray intensity.

3·1. *Rate of production of* ^{3}He *and Tritium by cosmic rays.* – In our initial publication [6] we calculated the rate of production of ^{3}He (as given in Table II), and of tritium [18] (as given in Table III) by the following method:

The meteorite was considered to be of infinite size and the production rate was determined at the position of the transition maximum, about 2 to 15 cm deep for an iron meteorite. A quantitative calculation was carried out using

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(16) L. A. CURRIE, W. F. LIBBY and R. L. WOLFGANG: *Phys. Rev.*, **101**, 1557 (1956).
(17) F. BEGEMANN, J. GEISS and D. C. HESS: *Phys. Rev.*, **107**, 540 (1957).
(18) S. F. SINGER: *Phys. Rev.*, **90**, 168 (1953).

TABLE II. – *Helium-3 production rate in meteorites.*

Production rate cm^3 (STP) per g per year	Location (relative to preatmospheric skin)	Method
$1.5 \cdot 10^{-14}$	At transition maximum	Calculated from cosmic-ray data and evaporation theory [a]
$\sim 0.9 \cdot 10^{-14}$	Unknown, but probably close to transition maximum	Calculated from measured tritium activity [b]

(*a*) Reference (6).
(*b*) Reference (17).

TABLE III. – *Tritium production rates in meteorites.*

Production rate (atoms/g/s)	Activity (counts/min/g)	Location	Method
$8 \cdot 10^{-3}$	0.50	At maximum near skin of very large meteorite.	Calculated from cosmic-ray data and evaporation theory [a].
$3.5 \cdot 10^{-4}$	0.02	Reduces to large meteorite case by dividing by a geometric factor of 4.	Calculated from experiments using 2 GeV protons [b].
$8.8 \cdot 10^{-4}$	0.05	At maximum near skin of a very large meteorite.	Calculated from experiments with 430 MeV and 2 GeV protons [c].
$4.7 \cdot 10^{-3}$	0.28	Unknown.	Measured [d].

(*a*) Reference (18).
(*b*) Reference (15).
(*c*) Reference (16).
(*d*) Reference (17).

the present cosmic-ray flux near the earth, taking into account the transition effect due to low-energy cosmic-ray secondaries, and including also the interaction of π-mesons which would not decay but would produce additional nuclear disintegrations. This calculation was checked by a more empirical approach in which we analyzed a large sample of nuclear evaporations, about 12 000 stars obtained by the Bristol group, as to their excitation energy. Then, by applying the results of the evaporation theory (19), the rate of production of ^3He

(19) K. J. LE COUTEUR: *Proc. Phys. Soc. London*, A **63**, 259 (1950).

and tritium was determined. For the distribution of excitation energies under consideration about 65% of the production of ^{3}He is by way of tritium, while the rest is directly produced. More recently, a rate of production of tritium has been calculated based on experiments in accelerators with 430 MeV and 2 GeV protons ([15,16]). These values have been consistently lower than ours ([18]) (see Table III) without any obvious reason for the discrepancy. We cannot say for certain that the results of BEGEMANN *et al.* ([17]) check our calculations; we do not know the position of their sample with respect to the meteorite boundary. Furthermore, there is some difference between stone and iron in the types of nuclear evaporations produced. However, their experimental determination of the cosmic-ray production rate for tritium seems to favor our higher value; in any case their value should be less than the value at the transition maximum. The discrepancy between our value and the value derived from the cyclotron experiments may perhaps be due to the fact that the excitation energy of the evaporating nuclei is much higher in the cosmic-ray case than for the cyclotron experiments because of the higher mean energy of cosmic rays. A 2 GeV proton will tend to give its energy to a knock-on nucleon and furnish only about 50 MeV for nuclear excitation ([20]). Particles of the order of 10 GeV, however, would create many mesons in a single interaction, but in a heavy nucleus a substantial fraction of these mesons are immediately absorbed. In each reabsorption approximately 50 MeV will be given to the nucleus. The process of meson production and reabsorption within the same nucleus is a means of making available a large amount of energy for nuclear excitation. This hypothesis, if true, would mean that the cross-section for tritium and ^{3}He production should rise beyond 2 GeV. It would therefore be very important to continue the experimental work beyond this energy.

3·2. *Distribution of meteorites ages.* – We can now inquire why the meteorite breakup times should show a clustering near 300 million years. Even though the sample is very sparse (6 meteorites) and the uncertainty in age rather high, the probability of such a close coincidence is very low and points to an underlying reason.

One can, of course, assume that *all* meteorites were created 300 million years ago but this seems hardly probable. A more likely explanation is one based on the work of ÖPIK who has shown ([21]) that asteroidal bodies crossing the earth's orbit have a mean life of the order of 100 million years (Table IV). Therefore, the probability that meteorites could have survived for much more than 300 million years is rather small.

Starting with a few planetoids the rate of creation of meteorites probably first rose exponentially, as the numbers of fragments increased and therefore

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([20]) I am indebted to Professor G. PUPPI for discussion on this point.

([21]) E. J. ÖPIK: *Proc. Roy. Irish Acad.*, A 54, 165 (1951); *Contr. Armagh Obs.*, no. 6.

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TABLE IV. – *Lifetimes of objects having multiple crossing with inner planets* [a].

Object	Apollo	Adonis	Hermes	Icarus	Encke's comet	Geminids (meteor shower)
	(←	asteroids		→)		
Inclination	6.4°	1.5°	4.7°	23.2°	12.5°	23.5°
Eccentricity	0.57	0.78	0.47	0.83	0.85	0.90
Mean life, 10^6 yr	72	72	42	170	270	250

(*a*) From reference ([21]).

also the probability of collisions. At the same time, the disrupting collisions also placed the fragments into eccentric orbits so that they were rapidly swept up by the planets. This constant attrition of meteorites by capture probably slowed the rising curve and let it reach a maximum. From then on, the number of new meteorites created by fragmentation must have dropped radically. At the present time, fragmentations of large bodies probably occur only occasionally, and can no longer be described by a continuous curve (on a time scale of 100 million years) but rather as random events. In this way the existence of a peak in the vicinity of 300 million years can be interpreted quite reasonably as a particularly effective creation of meteorites in the last 500 million years. Earlier creations would have been attenuated so severely that they are no longer incident on the earth.

We can say very little about the dispersion of the meteorite ages. Because of the reasons given earlier, we do not know how to find unambiguously the age of meteorites having a very small ^{3}He content. They could be regarded as having been created in much more recent times, say in the last 50 million years, without contradicting the present experimental evidence. The number of meteorites reported ([8]) to have a low ^{3}He content consistent with such an age is approximately 10, while the number having no detectable helium content is about 5.

3·3. *Remarks on the prehistoric cosmic-ray intensity.* – We can make use of the good accord between Öpik's calculations of lifetime and the meteorite ages determined from the cosmic-ray method to make some definite remarks about the prehistoric cosmic-ray intensity. It gives one confidence that the cosmic-ray intensity has not varied greatly in prehistoric times. While we cannot, of course, exclude fluctuations of any magnitude, the average value probably was within a factor of 2 of its present value.

The analysis of MARTIN ([22]) according to which the cosmic-ray intensity must have been 3 or more times its present value, is questionable ([11]) since

([22]) G. R. MARTIN: *Geochim. et Cosmochim. Acta*, **3**, 288 (1953).

it is based on a solidification age for the planetoids of 100 million years obtained with the use of the ^{4}He-uranium method ([8,23]).

A more precise statement would be possible if we knew the exact orbital characteristics of the meteorites under consideration. Using Öpik's theory ([21]), it would then be possible to calculate the probable lifetime of the object against capture by the inner planets. Assuming that the orbit resulted when the meteorite was created, namely that the collision which creates the meteorite also makes the orbit eccentric enough to intersect the earth, we would then obtain the precise exposure time to cosmic rays.

The use, in addition to tritium, of other radioactive nuclides, e.g., ^{53}Mn with a suggested half-life of $\sim 10^{6}$ years ([24]), would help to check the constancy of cosmic-ray flux. This fascinating aspect of meteorite and cosmic-ray science is just beginning.

([23]) Neither can one conclude that the cosmic ray intensity has been 1/10 to 1/20 its present value just because the ^{3}H : ^{3}He age is so much less than the ^{40}A : ^{40}K age ([17]). The two « ages » refer to different physical events.

([24]) R. K. Sheline and J. E. Hooper: *Nature*, **179**, 85 (1957).

Note added:

Our attention has been called to a paper just published by E. L. Fireman and D. Schwarzer (*Geochim. Cosmochim. Acta*, **11**, 252 (1957)). Their paper was submitted in August 1956, and represents therefore the first successful application of the ^{3}H-^{3}He method to the measurement of the cosmic ray age of meteorites. The authors have examined two recently fallen iron meteorites and obtain the following ages: Para de Minas ($1700 \cdot 10^{6}$ y.), and Norfolk ($900 \cdot 10^{6}$ y.); both are much higher than the ages shown in Table I. This difference, however, only serves to reinforce our hypothesis (see point 2 of Discussion) that planetary breakups occurred frequently throughout the history of the solar system. We would expect that the observed distribution of meteorite cosmic ray ages is the product of two functions: (i) the distribution of break-up dates (showing a peak early in the history of the solar system); and (ii) the exponential sweep-up factor (with mean life $(100 \div 200) \cdot 10^{6}$ y.).

RIASSUNTO (*)

La nascita di un meteorite coincide con la frantumazione del corpo che gli dà origine nonchè con l'inizio della sua esposizione ai raggi cosmici. Tale esposizione continuata crea attraverso evaporazioni nucleari quantitativi esattamente calcolabili di ^{3}He, circa $\frac{2}{3}$ dei quali passando per il tritio. La data di nascita del meteorite può quindi

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(*) *Traduzione a cura della Redazione.*

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essere desunta dal contenuto di ^{3}He; o meglio ancora dal contenuto di ^{3}He e di tritio misurati nello stesso campione. Tuttavia, quest'ultimo metodo è di applicabilità limitata. I due metodi danno risultati in buon accordo fra di loro; sei meteoriti esaminati furono tutti trovati vecchi di circa 300 milioni di anni. Tale accordo convalida i calcoli del tasso di produzione dell' ^{3}He e porta a qualche considerazione sul processo d'eccitazione nucleare. L'età riscontrata è anche in buon accordo coi calcoli di Öpik per la vita media dei corpi meteoritici rispetto alla cattura da parte dei pianeti interni. Da questo accordo si possono trarre conclusioni sul meccanismo di creazione dei meteoriti e sulla costanza relativa dell'intensità della radiazione cosmica negli ultimi 300 milioni di anni.

Associated Production of Σ^0 and $K^+_{\mu 2}$.

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(ricevuto il 5 Febbraio 1958)

Summary. — A direct evidence for the associated production of a Σ^0 along with a $K^+_{\mu 2}$ has been obtained. The most interesting feature of the picture is that, besides showing the decays of a $\Lambda^0(\to \pi^- + p)$ and $K^+_\mu\,(\to \mu^+ + \nu)$ it shows the materialization of the photon resulting from $\Sigma^0 \to \Lambda^0 + \gamma$. The values of the energy of the emitted photon as derived from the decay dynamics and also from the energies of the soft electrons produced by it agree remarkably well. An overall energy balance is obtained for the whole event which is represented by the following reaction

$$\begin{array}{llll}
\pi^+ + n \to & \Sigma^0 + & K_\mu & \\
 & \downarrow & \searrow & \mu^+ + \nu \\
 & \Lambda^0 + \gamma & & \\
 & \downarrow \quad \searrow & \text{soft shower} & \\
 & \pi^- + p & &
\end{array}$$

One of the consequences of the Gell-Mann (1) model of displaced charge multiplates for the strange particles is the prediction of a Σ-triplet (Σ^+, Σ^0, Σ^-) of which the charged counterparts (Σ^+, Σ^-) have been observed in various laboratories. The possible existence of the Σ^0 has been inferred from the dynamics of the associated production of Λ^0-θ^0 pairs by FOWLER *et al.* (2), WALKER (3) in the diffusion cloud chamber and later by TRILLING and LEIGHTON (4) in the propane bubble chanber. Recently direct evidence for the existence of Σ^0

(1) M. GELL-MANN: *Suppl. Nuovo Cimento*, **4**, 848 (1956).
(2) W. B. FOWLER, R. P. SHUTT *et al.*: *Phys. Rev.*, **91**, 1237 (1953).
(3) W. D. WALKER: *Phys. Rev.*, **98**, 1407 (1955).
(4) G. H. TRILLING and R. B. LEIGHTON: *Phys. Rev.*, **104**, 1703 (1956).

and its decay into Λ^0 and γ has been obtained by PLANO *et al.* (5) in a propane bubble chamber exposed to 1.15 GeV pion beam. In the three cases obtained by these authors the Σ^0 was produced along with a θ^0.

We reproduce in Fig. 1 a cloud chamber picture taken at Darjeeling (2 200 m) in which the evidence of the double production of a Σ^0 along with a $K^+_{\mu 2}$ is as direct and convincing as those obtained by PLANO *et al.* (5) since we also observe the materialization of the photon arising from the decay of the Σ^0 over and above the decay of the resulting Λ^0 into a pion and a proton. In our case, however, the Σ^0 has been produced associated with a $K^+_{\mu 2}$ whose decay products are also identified in the picture.

A primary particle PI interacts at I in the lead plate (11.2 g/cm²) and produces the particle IO' which stops and decays at O' into a muon $O'S_1$ (in the backward direction) of visible range greater than 92 g of Pb equivalent. The plates in the chamber are alternately 5.5 g of Cu and 11.2 g of Pb beginning and ending with Cu. Since the ionization of the particle $O'S_1$ in the last gap is appreciably greater than minimum we put its range $R_\mu = (97 \pm 5)$ g of Pb equivalent. This agrees remarkably well with the range of muons from $K^+_{\mu 2}$ as obtained by the Ecole Polytechnic group and the G-stack collaboration (5). Moreover the range of the particle IO' is (38 ± 3) g of Pb after it ionizes between 1.5 to 2 times minimum and therefore it is easily identified as a K-particle of mass between 900 to 1 000 times the electron mass. Thus $IO'S_1$ actually represents a $K_{\mu 2}$ event in which the K-particle had a momentum of (369 ± 10) MeV/c at the point of production. Taking the mass of $K_{\mu 2}$ to be 966 m_e its total energy is found to be (615 ± 6) MeV at the time of emission, where the statistical error shown is only due to the uncertainty in the total range.

At O, just below the interaction, we find a V^0-event whose secondaries OS_2 and OS_3 are definitely identified to be a pion and a proton from the following considerations. The secondary OS_2 has only a range of (15 ± 6) g of Pb from the point of ejection and ionizes about three times minimum before stopping in 11.2 g of Pb. Hence it is surely an L-meson. Besides this, the particle OS_2 suffers a large angle scattering in the only plate (5.5 g of Cu) through which it passes and hence it is taken to be a pion. The other secondary OS_3 has a visible range greater than 66 g of Pb equivalent and ionizes four to six times minimum in the last gap. Its range is more than 55 g of Pb after it ionizes more than two times minimum and this fact puts its mass much above the K-particle mass and very near to that of the proton. This secondary is therefore identified as a proton of total range (73 ± 6) g of Pb at the time

(5) J. CRUSSARD, V. FOUCHÉ, G. KAYAS, L. LEPRINCE-RINGUET, D. MORELLET, F. RENARD and J. TREMBLEY: *Suppl. Nuovo Cimento*, **4**, 373 (1956); G-STACK COLLABORATION: *Suppl. Nuovo Cimento*, **4**, 398 (1956).

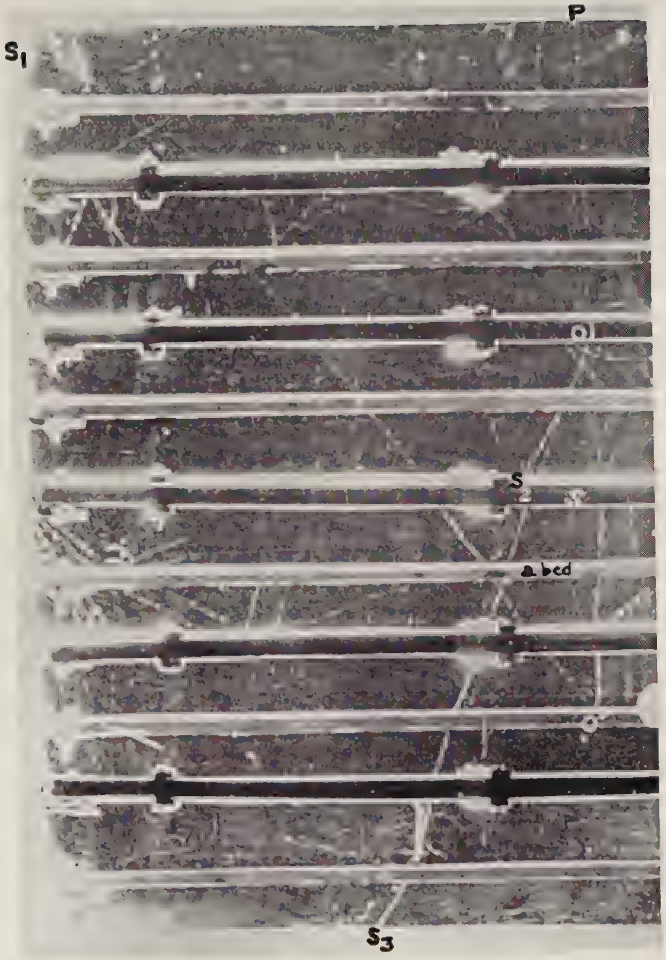

Fig. 1. (Left View).

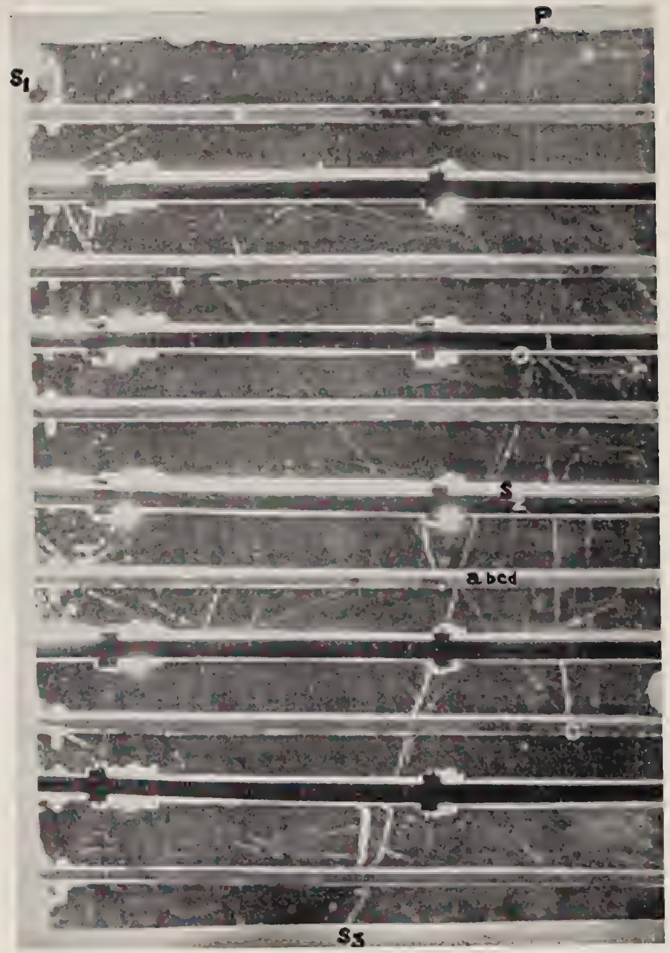

Fig. 1. (Right View).

Fig. 1. — Associated production of a Σ^0 and a K_μ. The primary particle PI interacts at I in a lead plate and produces a Σ^0 and K_μ^+. The latter stops and decays at O' into the muon $O'S_1$. The former decays into a Λ^0 and a γ; the Λ^0 then further decays into the pion OS_2 and proton OS_3 and the γ produces the cascade shower a, b, c, d.

of emission and has therefore a momentum of (709^{+23}_{-19}) MeV/c and a total energy of (1174 ± 13) MeV. Having identified the decay products as a pion and a proton we take the V^0 event at O to be actually the decay of a Λ^0 whose momentum and energy are accurately determined from the total energies of its decay products and the known mass 1113 MeV of the Λ^0. They are (774 ± 28) MeV/c and (1356 ± 17) MeV respectively. As the secondaries OS_2 and OS_3 are observed in a plane almost in the line of sight of both the right and left views, the angle between them is difficult to measure. The difficulty is all the more enhanced by the large angle scattering of both the secondaries in the very first plate they pass through (see Fig. 2 C). Nevertheless the direction of the Λ^0 and its two secondaries can be calculated from their known momenta and the dynamics of a Λ^0 decay. We have therefore determined as accurately as possible the point O, the direction of the pion OS_2 in space and the Λ^0-decay plane and then constructed the direction of the Λ^0 and proton as required for a Λ^0-decay. The result is shown in Fig. 2C. The directions of the primary PI and the K_μ-particle IO' do not change abruptly and their plane is fairly correctly determined. It is found in the stereo projection that the apex O of the Λ^0-decay does not lie in the plane determined by PI and IO'. In fact the true direction of the Λ^0 as determined previously makes an angle of $(6 \pm 2)°$ with the plane PIO' pointing thereby to the possible emission of a Σ^0 as an intermediate particle just as in the cases of the Λ^0-θ^0 pairs observed in the Brookhaven diffusion chamber and the Berkeley propane bubble chamber. An exactly similar double production of a Λ^0 and $K^+_{\mu 2}$ observed by BARKER *et al.* [6] does not however exhibit any non-coplanarity in the Λ^0-K_μ plane. The range, momenta, and energies of the various secondaries and also the angles between them are given in Table I.

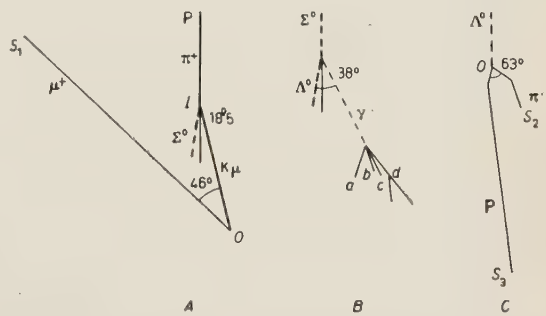

Fig. 2. – Schematic drawing of the whole sequence of events of Fig. 1 with connected non-ionizing links (dashed lines), and various angles in space. « A » denotes the production of Σ^0 and K^+_μ along with the latter's decay. « B » shows the disintegration of Σ^0 into Λ^0 and γ with the latter's materialization into four electrons a, b, c, d. « C » depicts the decay of the Λ^0 and the large angle scatterings of both of its decay products.

([6]) K. H. BARKER, M. S. COATES and B. R. FRENCH: *Suppl. Nuovo Cimento*, **4**, 319 (1956).

TABLE I. – *Range, momenta and energies of the various secondaries and the various angles between them.*

Particle	Identified as	Observed range in g/cm² of Pb	Momentum in MeV/c	Total energy in MeV	Angle between
IO'	K_μ^+	38 ± 3	369 ± 10	615 ± 6	
O'S$_1$	μ^+	97 ± 5	225 ± 11	249 ± 10	$\widehat{K\text{-}\mu} = 46° \pm 3$
OS$_2$	π^-	15 ± 6	117^{+13}_{-18}	182 ± 11	$\widehat{\pi^-\text{-p}} = 63° \pm 3$ (calculated value)
OS$_3$	p	73 ± 6	709^{+23}_{-19}	1174 ± 13	
IO	Λ^0	(calculated values)	774 ± 28	1356 ± 17	
IS	γ		114 ± 15	114 ± 15	$\widehat{\Lambda^0\text{-}\gamma} = 38° \pm 4$
	Σ^0	(calculated values)	867 ± 31	1470 ± 22	$\widehat{\pi^+\text{-}\Sigma^0} = 5.5° \pm 1.5$ (calculated value)
PI	π^+		1221 ± 38	1229 ± 40	$\widehat{\pi^+\text{-}K_\mu} = 13° \pm 2$

We now come to the most interesting feature of the picture reproduced in Fig. 1. It is the presence of a photon initiated cascade of four electrons *a b c d* of which *a b c* stop in 5.5 g of Cu and « *d* » produces another pair of electrons stopping in the next plate (11.2 g of Pb). The apex of this cascade shower when joined in space to the point of interaction *I* gives a direction which is well within the core of the electron shower and represents, in our opinion, the direction of the photon emitted in the decay process,

$$\Sigma^0 \to \Lambda^0 + \gamma \; ... \; . \tag{1}$$

We shall now try to make a quantitative estimate of the energy of the ejected photon from (i) the dynamics of the decay mode (1) and also from (ii) the nature of the soft shower produced by it. Firstly we take the Σ^0 and Λ^0 masses to be 1.187 GeV and 1.113 GeV respectively and conserving momenta and energies in (1) get the energy E_γ of the emitted photon as

$$E_\gamma = \frac{0.0847}{E_{\Lambda^0} - p_{\Lambda^0} \cos\theta} \text{ GeV} \; ... \; . \tag{2}$$

where E_{Λ^0} and p_{Λ^0} are the energy an momentum of the emitted Λ^0 in GeV and GeV/c and θ the angle between the Λ^0 and the photon. The direction of the photon *IS* as determined by joining the apex of *a b c d* to *I* in the stereo-projection, makes an angle of $(38 \pm 4)°$ with the calculated direction of Λ^0 as depicted in Fig. 2*B*. This value of θ combined with the known values of E_{Λ^0} and p_{Λ^0} from Table I gives $E_\gamma = (114 \pm 15)$ MeV. Secondly the pho-

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ton apparently traverses a distance of 2.3 radiation lengths (5.5 g of Cu and 11.2 g of Pb) before giving a maximum number of four electrons and has therefore and average energy of 125 MeV according to the cascade theory. Thus we find a remarkable agreement between the dynamically required value of the energy of the emitted photon and that obtained from the nature of the soft shower initiated by it according to the cascade theory.

We therefore conclude that a Σ^0 was originally emitted at I along with the K^+_μ and the Σ^0 immediately (in $< 10^{-20}$ s) decayed into a Λ^0 and γ. A schematic diagram of the whole event with connected non-ionizing links (as explained above) and various angles in space is given in Fig. 2*A*, 2*B* and 2*C*. Fig. 2*A* gives the production process together with the K^+_μ decay event, Fig. 2*B* the decay of Σ^0 and the materialization of the emitted photon and Fig. 2*C* the decay of Λ^0 and the large angle scatterings of the resulting pion and proton.

Knowing the value of the energy of the emitted photon and also the energy of the Λ^0 from its decay products, the energy and consequently the momentum of the Σ^0 is obtained from its predicted mass (1187 MeV). So far we have not made any assumption as to the nature of the collision which has resulted in the associated pair and which has taken place in a lead nucleus. We notice that no other charged particle has emerged from the collision except the $K^+_{\mu 2}$. We therefore believe that a simple collision has occurred with a peripheral neutron and as a first approximation we balance the transverse momentum of the Σ^0 and K^+_μ assuming the neutron to be stationary in the laboratory system. The angle which the K^+_μ makes with the incident primary has been measured to be $(13 \pm 2)^\circ$ abd hence the angle of emission of the Σ^0 is found to be $(5.5 \pm 1.5)^\circ$. Conserving longitudinal momentum, the momentum of the incident particle is obtained as (1221 ± 38) MeV/c. In order to conserve strangeness and the number of baryons the incident particle has to be a positive pion so that the whole sequence of events is represented by the following equation:

$$
\begin{array}{ccccc}
\pi^+ + n \to & \Sigma^0 & + & K^+_\mu & \\
 & \downarrow & & & \searrow \\
 & \Lambda^0 + \gamma & & & \mu^+ + \nu \\
 & \downarrow \quad \searrow & & & \\
 & \pi^- + p \quad \text{soft shower} & & &
\end{array} \tag{3}
$$

If we now balance the total energy on the two sides of equation (3), we find an unbalance of (83 ± 46) MeV which is only $(4 \pm 2)\%$ of the total energy of 2085 MeV of the Σ^0 and K^+_μ This unbalance is quite small and can be neglected. Even if it is real, it can be accounted for by the Fermi momentum of the interacting neutron inside the nucleus which may be as high as 270 MeV/c for a lead nucleus.

We therefore conclude that this picture shows an event in which we find direct evidence for the production of an associated pair of Σ^0 and K_μ^+ together with their subsequent decays as represented by equation (3). It is also interesting to note that if we make a relativistic transformation of the production angles to the centre of mass system, we find that the Σ^0 has been emitted in the forward direction and the K_μ^+ in the backward direction. This agrees well with the observation of LEIGHTON and TRILLING [4] in the production Σ^--K pairs.

* * *

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RIASSUNTO (*)

Si è ottenuta una prova diretta della produzione associata di un Σ^0 con un $K_{\mu 2}^+$. La caratteristica più interessante del quadro è che oltre a mostrare i decadimenti di un $\Lambda^0(\to \pi^- + p)$ e di un $K_\mu^+(\to \mu^+ + \nu)$ mostra la materializzazione del fotone risultante dalla reazione $\Sigma^0 \to \Lambda^0 + \gamma$. I valori dell'energia del fotone emesso, quale risulta dalla dinamica del decadimento nonchè dalle energie degli elettroni molli prodotti si accordano assai bene. Si ricava il bilancio energetico totale dell'intero evento che è rappresentato dalla reazione seguente:

$$\begin{tikzcd}[column sep=small]
\pi^+ + n \to & \Sigma^0 \arrow[d] & + K_\mu \arrow[dr] & \\
& \Lambda^0 + \gamma \arrow[d] \arrow[dr] & & \mu^+ + \nu \\
& \pi^- + p & \text{sciame molle} &
\end{tikzcd}$$

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. VIII, N. 4 16 Maggio 1958

The Solutions of the Low Equation - II.

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(ricevuto il 17 Febbraio 1958)

Summary. — The Low equation for a non-relativistic factorizable potential is investigated. The familiar solution for $h(k)$, where $h(k)$ is the scattering amplitude, is obtained directly as a Low equation by proceeding from the eigenstates of the total Hamiltonian to the eigenstates of the free Hamiltonian. The relation between the manifold of solutions of the Low equation for $h(k)$ and the spectrum of the free Hamiltonian is immediate. The zeros of $h(k)$ are the bound state energies of the free Hamiltonian.

1. – Introduction.

In a previous paper (¹) we discussed the relation between the manifold of solutions of the Low equation, discovered by Castillejo, Dalitz and Dyson (²), and the existence of «hidden structure», that is to say bound states of the free Hamiltonian. In particular the Lee-type model of Dyson (³) and Norton and Klein (⁴) was discussed in detail. In this paper we wish to consider the simple case of a non-relativistic factorizable potential which has been previously discussed by Haag (⁵). He showed how one can construct a Low

(¹) D. B. Fairlie and J. C. Polkinghorne: *Nuovo Cimento*, **8**, 345 (1958). Referred to as I.

(²) L. Castillejo, R. H. Dalitz and F. J. Dyson: *Phys. Rev.*, **101**, 453 (1956).

(³) F. J. Dyson: *Phys. Rev.*, **106**, 157 (1957).

(⁴) R. E. Norton and A. Klein: *Phys. Rev.*, **109**, 584 (1958).

(⁵) R. Haag: *Nuovo Cimento*, **5**, 203 (1957).

equation for this model starting from an expression for the eigenstates of the total Hamiltonian in terms of the eigenstates of the free Hamiltonian. The resulting equation for the scattering amplitude $h(k)$ can be solved by the trick ([6]) of considering $H(k)=1/h(k)$ and obtaining from mathematical considerations the equation satisfied by $H(k)$. The manifold of solutions arises from the fact that $h(k)$ can be given an arbitrary number of zeros which produce an arbitrary number of poles of $H(k)$ and a consequent arbitrariness in the expression for $H(k)$.

In this paper we adopt a converse procedure and use an expression for the eigenstates of the free Hamiltonian in terms of the eigenstates of the total Hamiltonian. This yields a Low-type equation which turns out to be just the equation for $H(k)$ previously obtained by indirect mathematical arguments. The way in which we obtain this equation makes it clear that for this model the poles of $H(k)$, or equivalently the zeros of $h(k)$, correspond to the bound states of the free Hamiltonian.

2. – The reciprocal Low equation.

We consider a non-relativistic system with H_0 the free Hamiltonian; $H=H_0+V$ the total Hamiltonian; $|\overset{+}{k}\rangle$ the set of eigenstates of H with eigenvalues E_k and satisfying the condition that

$$\lim_{t\to\infty} \exp[iH_0 t]\exp[-iHt]|\overset{+}{k}\rangle = |k\rangle . \tag{1}$$

where $|k\rangle$ are eigenstates of H_0 with eigenvalues E_k. We shall suppose that in addition to the $|k\rangle$, H_0 possesses a set of bound-state eigenstates $|\alpha\rangle$ of energies E_α. The complete set of eigenstates of H_0, both $|k\rangle$ and $|\alpha\rangle$, will be denoted by $|l\rangle$ and we have

$$\sum_l |l\rangle\langle l| = 1 . \tag{2}$$

The usual way of setting up a Low equation for this system is to consider the expression

$$|\overset{+}{k}\rangle = |k\rangle - \frac{1}{H-E_k+i\varepsilon} V|k\rangle , \tag{3}$$

([6]) G. F. Chew and F. E. Low: *Phys. Rev.*, **101**, 1570 (1958).

which gives the eigenstates of H in terms of the eigenstates of H_0 and leads to the equation

$$\langle \overset{+}{k}' | V | k \rangle = - \sum_{k''} \frac{\langle \overset{+}{k}'' | V | k' \rangle^* \langle \overset{+}{k}'' | V | k \rangle}{E_{k''} - E_{k'} - i\varepsilon} + \langle k' | V | k \rangle . \tag{4}$$

If the potential V is factorizable so that we may write

$$\langle l' | V | l \rangle = A\, u^*(l') u(l) , \tag{5}$$

then (4) becomes the Low equation

$$h(k') = - \sum_{k'} \frac{|u(k'')|^2 |h(k'')|^2}{E_{k''} - E_{k'} - i\varepsilon} + A , \tag{6}$$

where $h(k')$ is defined by

$$\langle \overset{+}{k}' | V | k \rangle = \frac{t(k') u(k)}{u(k')} , \tag{7}$$

$$h(k') = \frac{t(k')}{|u(k')|^2} . \tag{8}$$

The summation on the right hand side of (5) represents a combined sum and integral over the energy spectrum of H.

We reverse this procedure and consider $H_0 = H - V$. Then for an eigenstate $|k\rangle$ of H_0 belonging to the continuum we have

$$|k\rangle = |\overset{+}{k}\rangle + \frac{1}{H_0 - E_k + i\varepsilon} V |\overset{+}{k}\rangle , \tag{9}$$

as the analogue of (3). Since the states $|k\rangle$ and $|\overset{+}{k}\rangle$ coincide at $t = \infty$ the choice of the sign of $i\varepsilon$ in (9) is the correct one. Equation (9) leads to

$$\langle k | V | \overset{+}{k}' \rangle = \langle \overset{+}{k} | V | \overset{+}{k}' \rangle + \sum_l \frac{\langle \overset{+}{k} | V | l \rangle \langle l | V | \overset{+}{k}' \rangle}{E_l - E_k - i\varepsilon} . \tag{10}$$

This equation is equation (16) of I derived from a different view point, and we shall now use it to solve the Low equation. From equations (5) and (7) and the identity

$$\langle \overset{+}{k}' | V | k \rangle = \sum_l \langle \overset{+}{k}' | l \rangle \langle l | V | k \rangle , \tag{11}$$

we deduce that

$$\sum_l \langle \overset{+}{k'}|l\rangle u^*(l) = \frac{t(k')}{Au(k')} . \tag{12}$$

We have the identity

$$\langle \overset{+}{k}|V|\overset{+}{k'}\rangle = \sum_l \sum_{l'} \langle \overset{+}{k}|l\rangle\langle l|V|l'\rangle\langle l'|\overset{+}{k'}\rangle , \tag{13}$$

and from (5), (12) and (13) it follows that

$$\langle \overset{+}{k}|V|\overset{+}{k'}\rangle = \frac{t(k)t^*(k')}{Au(k)u^*(k')} . \tag{14}$$

Substitution from (7), (8) and (14) into (10) yields

$$\frac{1}{h(k)} = \frac{1}{A} + \sum_l \frac{|u(l)|^2}{E_l - E_k - i\varepsilon} . \tag{15}$$

The summation on the right hand side of (15) is over the energy spectrum of H_0. It is an integral over the continuum states $|k\rangle$ and a sum over the bound states $|\alpha\rangle$. That is

$$H(k) = \frac{1}{h(k)} = \frac{1}{A} + \sum_\alpha \frac{|u(\alpha)|^2}{E_\alpha - E_k - i\varepsilon} + \int \mathrm{d}k' \frac{|u(k')|^2}{E_{k'} - E_k - i\varepsilon} . \tag{16}$$

This is just the form of the solution of the Low equation (6) corresponding to the generalized R-function

$$R(k) = \sum_\alpha \frac{|u(\alpha)|^2}{E_\alpha - E_k - i\varepsilon} , \tag{17}$$

in which $H(k)$ has poles at $E_k = E_\alpha$.

3. – Discussion.

For this model we have given a quasi-physical explanation of the trick used for solving the Low equation. This explanation has made it clear how the properties of the solution correspond to the spectral properties of H_0, the free Hamiltonian. In fact the zeros of the scattering amplitude occur at the energies of the bound states of H_0.

There is always an intimate connection between the zeros of $h(k)$ and the bound states of H_0, though this relation is not always quite as direct as that

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discussed here. In I we considered the Dyson model, which differs from the model treated here in that equation (5) is only true if one state is a continuum state and the other a bound state. There the relation between the « hidden structure » and the generalized R-function was of a different, but equally simple, type.

RIASSUNTO (*)

Si studia l'equazione di Low per un potenziale fattorizzabile non relativistico. La soluzione familiare per $h(k)$, dove $h(k)$ è l'ampiezza di scattering si ottiene direttamente come equazione di Low passando dagli autostati dell'hamiltoniana totale agli autostati dell'hamiltoniana libera. La relazione tra la famiglia di soluzioni dell'equazione di Low per $h(k)$ e lo spettro dell'hamiltoniana libera è immediata. Gli zeri di $h(k)$ sono le energie degli stati legati dell'hamiltoniana libera.

(*) *Traduzione a cura della Redazione.*

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The High Energy Limit of the Potential Scattering.
II. Relativistic Kinematics.

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(ricevuto il 21 Febbraio 1958)

Summary. — The asymptotic expansions of the scattering amplitude at high energy obtained previously for a Schrödinger equation of motion, are extended to the cases of a Dirac and Klein-Gordon particle. The method followed here is based on the extension of the so called Gel'fand and Levitan operator which gives a uniform connection of two complete bases in the Hilbert space. A compact expression for the general term of the expansion is found. This allows its evaluation by means of recurrence relations.

1. – Introduction.

We wish to consider in this paper the same type of problem which we have already treated for the scattering of a non-relativistic scalar particle by an external field [1]. The same restriction of a statical spherical symmetrical potential will be maintained, but we consider particles of spin 0 and $\frac{1}{2}$ moving with velocity near to the limit value of the light velocity c.

We deduce some expansions of the phase shifts relative to a fixed angular momentum and parity, in powers of $(1-\beta^2)$, where $\beta = v/c$ is the ratio of the particle's velocity v and c.

The mathematical tool used in order to arrive at the expansions is closely related to that of the Schrödinger case [1]. Indeed, we start from an integral operator O which maps the waves Φ_0 distorted by an external field V_0 to

(1) M. VERDE: *Nuovo Cimento*, **6**, 340 (1957).

the corresponding waves Φ distorted by a different external field V. O has the peculiar feature of beeing independent of the energy. This is particularly suited to the study of the behaviour of Φ for large E, when the corresponding behaviour of Φ_0 is known.

The same study done with the mapping of Φ_0 into Φ by means of the familiar operator, which contains the appropriate Green's function G of the Φ_0's space, is much less powerful because G is energy dependent.

The operator O corresponding to the cases treated in this paper has not been given before (2) and it is presented here without proof. Since its deduction requires some extended considerations, and it is especially important in connection with the inversion problem of providing the potential when the spectral function is assigned, it forms the subject of a separated investigation (3).

2. – Potential scattering of a Dirac particle in a central field.

The Dirac equation of motion for a partial wave of angular momentum j and fixed parity reduces, as it is well known, to the following two component wave-equation

$$\left\{-i\sigma_2\frac{\partial}{\partial x}+m\sigma_3+\frac{j+\frac{1}{2}}{x}\,\sigma_1+V(x)\right\}\varphi(E,x)=E\varphi(E,x)\,, \tag{1.1}$$

where σ_k are the familiar Pauli matrices.

It is convenient in the following to work with an equivalent form of Eq. (1.1) performing a unitary transformation by means of the operator

$$U=\exp\left[i\sigma_2 Q\right]\qquad\text{with}\qquad \frac{\mathrm{d}Q}{\mathrm{d}x}=V(x)\,.$$

In the new frame of reference Eq. (1.1) becomes:

$$\left(-i\sigma_2\frac{\partial}{\partial x}+\Omega\right)\Phi(E,x)=E\Phi(E,x)\,, \tag{1.2}$$

where

$$\Omega=\exp\left[i\sigma_2 Q\right]\left(m\sigma_3+\frac{j+\frac{1}{2}}{x}\,\sigma_1\right)\exp\left[-i\sigma_2 Q\right]. \tag{1.3}$$

We consider now the motion in a different potential V_0, which in part-

(2) There is a tentative done for the Klein-Gordon equation of motion, which is however incomplete in our opinion. See E. CORINALDESI: *Nuovo Cimento*, **11**, 468 (1954).

(3) M. VERDE: in course of publication.

icular may vanish. The Dirac equation corresponding to (1.2) will be

$$\left(-i\sigma_2\frac{\partial}{\partial x}+\Omega^{(0)}\right)\Phi^{(0)}(E,x)=E\Phi^{(0)}(E,x)\,. \tag{1.2$'$}$$

Choosing the Φ and $\Phi^{(0)}$ so that they have the same behaviour for $x=0$, namely

$$\Phi,\,\Phi^{(0)}\simeq\begin{pmatrix}x^{j+\frac{1}{2}}\\ 0\end{pmatrix}+0(x^{j+\frac{3}{2}})\,, \tag{1.4}$$

the completness relations for Φ and $\Phi^{(0)}$ will be written:

$$\int\limits_{-\infty}^{+\infty}\mathrm{d}\varrho(E)\Phi_\alpha(E,x)\Phi_\beta(E,x')=\delta_{\alpha\beta}(x-x')=\int\limits_{-\infty}^{+\infty}\mathrm{d}\varrho_0(E)\Phi^{(0)}_\alpha(E,x)\Phi^{(0)}_\beta(E,x')\,,$$

where $\varrho(E)$ and $\varrho_0(E)$ are the so called spectral functions. $\mathrm{d}\varrho(E)$, $\mathrm{d}\varrho_0(E)$ have singularities of the type of the Dirac functions $\delta(E-E_n)$, $-m\leqslant E_n\leqslant+m$, for the eigenvalues E_n corresponding to the bound states of Eq. (1.2) respectively Eq. (1.2)$'$.

In complete analogy to the Schrödinger case, there is a connection between the two bases Φ and $\Phi^{(0)}$ of the Hilbert space, of the type

$$\Phi(E,x)=(1+iS\sigma_2)\Phi^{(0)}(E,x)\,, \tag{1.5}$$

the matrix elements of S being

$$S_{\alpha\beta}(E)=\int\limits_{-\infty}^{+\infty}\mathrm{d}(\varrho-\varrho_0)\,\frac{\Phi_\alpha(E'x)\Phi^{(0)}_\beta(E'x)}{E-E'}\,. \tag{1.6}$$

This follows from the Gel'fand and Levitan equation (3) equivalent to (1.2) t.i.

$$\Phi(E,x)=\Phi^{(0)}(E,x)-\int\limits_{-\infty}^{+\infty}\mathrm{d}\big(\varrho(E')-\varrho_0(E')\big)\Phi(E',x)\int\limits_0^x\Phi^{(0)T}(E',y)\Phi^{(0)}(E',y)\,\mathrm{d}y\,, \tag{1.7}$$

and the continuity equation for $\Phi^{(0)}$ namely

$$-i\Phi^{(0)T}(E',x)\sigma_2\Phi^{(0)}(E,x)=(E-E')\int\limits_0^x\Phi^{(0)T}(E',y)\Phi^{(0)}(E,y)\,\mathrm{d}y\,.$$

The form Eq. (1.6) of the operator S is well suited to the study of the behaviour of Φ when E is large, since we have to evaluate the moments

$$K^{(n)}_{\alpha\beta}(x, x) = \int_{-\infty}^{+\infty} E^n \,\mathrm{d}(\varrho - \varrho_0) \Phi_\alpha(E, x) \Phi^{(0)}_\beta(E, x) ,$$

in those cases which allow an expansion in inverse powers of E of the same operator S.

It is now possible to establish simple recurrence relations to which the moments have to satisfy and that serve also to their evaluation in terms of the difference of the potentials $V(x) - V_0(x) = \delta V$. Putting:

$$K^{(n)}_{\alpha\beta}(x, y) = \int_{-\infty}^{+\infty} E^n \,\mathrm{d}(\varrho - \varrho_0) \Phi_\alpha(E, x) \Phi^{(0)}_\beta(E, y) ,$$

and operating on it with $i\sigma_2(\partial/\partial x)$, from Eq. (1.2) one obtains

$$i\sigma_2 \frac{\partial K^{(n)}(x, y)}{\partial x} = \Omega(x) K^{(n)}(x, y) - K^{(n+1)}(x, y) . \tag{1.8}$$

If one operates on the left of the same $K^n(x, y)$ with $i\sigma_2(\partial/\partial y)$, remembering Eq. (1.2′) we get also:

$$-\frac{\partial K^{(n)}(x, y)}{\partial y} i\sigma_2 = K^{(n)}(x, y) \Omega^{(0)}(y) - K^{(n+1)}(x, y) . \tag{1.8′}$$

Since σ_2 anticommute with Ω and $\Omega^{(0)}$ one deduces:

$$\begin{aligned} \frac{\mathrm{d}K^{(n)}(x, x)}{\mathrm{d}x} = \left(\frac{\partial K^{(n)}}{\partial x}\right)_{y=x} + \left(\frac{\partial K^{(n)}}{\partial x}\right)_{y=x} &= i\big(\Omega(x)\sigma_2 K^{(n)}(x, x) - K^{(n)}(x, x)\sigma_2 \Omega^{(0)}(x)\big) + \\ &+ i\big(\sigma_2 K^{(n+1)}(x, x) - K^{(n+1)}(x, x)\sigma_2\big) . \end{aligned}$$

This is the desired recurrence relation, that can be put in an equivalent but more convenient form introducing the operators

$$T^{(n)}_\pm(x) = K^{(n)}(x, x) \pm \sigma_2 K^{(n)}(x, x) \sigma_2 .$$

One obtains then:

$$\left\{ \begin{aligned} &\frac{\mathrm{d}T^{(n)}_+}{\mathrm{d}x} = i(\Omega\sigma_2 T^{(n)}_- + \sigma_2 T^{(n)}_+ \Omega^{(0)}) , \\ &2i\sigma_2 T^{(n)}_- = \frac{\mathrm{d}T^{(n-1)}_-}{\mathrm{d}x} - i(\Omega\sigma_2 T^{(n-1)}_+ - \sigma_2 T^{(n-1)}_+ \Omega^{(0)}) . \end{aligned} \right. \tag{1.9}$$

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Thus the knowledge of the first moment $T_-^{(0)}$ and the boundary condition for $x=0$ allows the determination of all other higher moments. It is easy to deduce from the Eq. (1.7) where only $K^{(0)}(x,y)$ appears, by a simple integration by parts that

$$i\sigma_2 T_-^{(0)} = \Omega(x) - \Omega^{(0)}(x)\,. \tag{1.10}$$

The boundary conditions Eq. (1.4) give

$$T_+^{(n)}(0) = -\,T_-^{(n)}(0)\sigma_3\,,$$

which is sufficient for the integration of (1.9). One deduces then from (1.10) and (1.9)

$$T_+^{(0)}(x) = \text{cost} = -\,T_-^{(0)}(0)\sigma_3 = (2j+1)i\sigma_2(\delta V)_{x=0}\,,$$

$$T_-^{(1)}(x) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(\Omega-\Omega^{(0)}) - (j+\tfrac{1}{2})(\delta V)_0(\Omega-\Omega^{(0)})i\sigma_2\,,$$

$$\begin{aligned} T_+^{(1)}(x) = m^2\!\int_0^x\!\delta V\,\mathrm{d}x + (j+\tfrac{1}{2})^2\left(\int_0^x\frac{(\delta V)'}{x}\,\mathrm{d}x - \frac{\delta V}{x}\right) + {}\\ + i\sigma_2\left\{\frac{1-\exp\left[2i\sigma_2(Q-Q^{(0)})\right]}{2}\left(m^2+\frac{(j+\frac{1}{2})^2}{x^2}\right) + {}\right.\\ \left. + \,\delta V_0\big((j+\tfrac{1}{2})(j+\tfrac{3}{2})V(0) - (j^2-\tfrac{1}{4})V_0(0)\big)\right\}, \end{aligned}$$

and so on.

We have assumed here that $(\delta V)'$ vanishes for $x=0$, otherwise the moment $T_+^{(1)}$ would diverge.

For the evaluation of the scattering amplitude we need the values of the $T^{(n)} = T^{(n)}(\infty)$ when $x=\infty$. The first moments calculated above are:

$$T_-^{(0)} = m(\exp\left[2i\sigma_2\lambda\right] - \exp\left[2i\sigma_2\lambda^{(0)}\right])\sigma_1\,,$$

$$T_+^{(0)} = (2j+1)(\delta V)_0 i\sigma_2\,,$$

$$T_-^{(1)} = (j+\tfrac{1}{2})(\delta V)_0 T_-^{(0)}\,,$$

$$\begin{aligned} T_+^{(1)} = m^2\delta\lambda + (j+\tfrac{1}{2})^2\!\int_0^\infty\!\frac{(\delta V)'}{x}\,\mathrm{d}x + m^2\sigma_2\exp\left[i\sigma_2\delta\lambda\right]\sin\delta\lambda + {}\\ + \big(\tfrac{1}{2}(j-\tfrac{1}{2})(\delta V)_0 + V(0)\big)T_+^{(0)}\,, \end{aligned}$$

where

$$\lambda = \int_0^\infty V(x)\,\mathrm{d}x\,,\qquad \lambda^{(0)} = \int_0^\infty V_0(x)\,\mathrm{d}x\,,\qquad \delta\lambda = \lambda - \lambda^{(0)}\,.$$

Remembering the unitary transformation which brings the φ's into the Φ's. we write

$$\varphi = (\exp\left[-i\sigma_2\,\delta Q\right] + \mathrm{i}\exp\left[-i\sigma_2 Q\right] S \exp\left[i\sigma_2 Q^{(0)}\right]\sigma_2)\varphi^{(0)}\,.$$

Since at large distances

$$\varphi^{(0)} \simeq \left(\frac{1-\sigma_3}{2} + \frac{E+m}{k}\,\frac{1+\sigma_3}{2}\right)\begin{pmatrix}\sin\left(kr - \dfrac{j+\frac{1}{2}}{2}\,\pi\right)\\[2ex] \cos\left(kr - \dfrac{j+\frac{1}{2}}{2}\,\pi\right)\end{pmatrix},$$

it is easy to derive for the tangent of the phase shift, the expansion

$$\text{(1.10)}\qquad \operatorname{tg}\delta_j^{(+)} = -\frac{k}{E}\operatorname{tg}\delta\lambda\left\{1 + \frac{1}{E^2}\left[m^2\left(\frac{1}{2} + \frac{\delta\lambda}{\sin 2\,\delta\lambda}\right) + \right.\right.$$
$$\left.\left. + (j+\tfrac{1}{2})^2\,\frac{1}{\sin 2\,\delta\lambda}\int\limits_0^\infty \frac{(\delta V)'}{x}\,\mathrm{d}x\right] + O\left(\frac{1}{E^4}\right)\right\}.$$

The phase shift $\delta_j^{(-)}$ corresponding to the same j but opposite parity in the same order of approximation results to be the same.

3. – Potential scattering of particles with spin zero.

If $\varphi(E, x)$ and $\varphi^{(0)}(E, x)$ are the stationary eigenstates of a Klein-Gordon particle in a central potential $V(x)$ respectively V_0, so that

$$\text{(2.1)}\qquad \left\{\begin{aligned} &\left\{(E-V)^2 - \mu^2 - \frac{l(l+1)}{x^2} + \frac{\partial^2}{\partial x^2}\right\}\varphi = 0\,,\\ &\left\{(E-V_0)^2 - \mu^2 - \frac{l(l+1)}{x^2} + \frac{\partial^2}{\partial x^2}\right\}\varphi^{(0)} = 0\,,\end{aligned}\right.$$

and we choose the normalization $\varphi, \varphi^{(0)} \simeq x^{l+1}$ when $x = 0$, there is again a uniform connection between the two complete bases of the Hilbert spaces expressed by the equation

$$\text{(2.2)}\qquad \varphi(E, x) = \varphi^{(0)}(E, x)\left[\cos\left(Q - Q^{(0)}\right) - \int\limits_{-\infty}^{+\infty}\mathrm{d}\sigma(E')\,\frac{\varphi(E', x)\varphi^{(0)\prime}(E', x)}{E - E'}\right] +$$
$$+ \varphi^{(0)\prime}(E, x)\int\limits_{-\infty}^{+\infty}\mathrm{d}\sigma(E')\,\frac{\varphi(E', x)\varphi^{(0)}(E', x)}{E - E'}\,,$$

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where the dashes indicate derivatives with respect to the radial variable x, and the other symbols are defined analogously to the spin $\frac{1}{2}$ case.

Namely

$$Q - Q^{(0)} = \int_0^x \delta V(x')\,\mathrm{d}x' , \qquad \mathrm{d}\sigma = \mathrm{d}\varrho - \mathrm{d}\varrho_0 ,$$

and

$$\int_{-\infty}^{+\infty} \mathrm{d}\varrho(E)\varphi_\alpha(E, x)\varphi_\gamma(E, x')(\sigma_1)_{\gamma\beta} = \delta_{\alpha\beta}(x - x') = \int_{-\infty}^{+\infty} \mathrm{d}\varrho_0(E)\varphi_\alpha^{(0)}(E, x)\varphi_\gamma^{(0)}(E, x')(\sigma_1)_{\gamma\beta} ,$$

where φ_α, $\varphi_\beta^{(0)}$ are two-components wave-functions

$$\varphi_\alpha(E, x) = \begin{pmatrix} \varphi(E, x) \\ \dfrac{E - V}{\mu}\,\varphi(E, x) \end{pmatrix} , \qquad \varphi_\alpha^{(0)}(E, x) = \begin{pmatrix} \varphi^{(0)}(E, x) \\ \dfrac{E - V_0}{\mu}\,\varphi^{(0)}(E, x) \end{pmatrix} .$$

Eq. (2.2) is an immediate consequence of the Gel'fand-Levitan form of the Klein-Gordon equation.

Indeed this form reads ([3]),

$$\varphi_\alpha(E, x) = \cos(Q - Q^{(0)})\varphi_\alpha^{(0)}(E, x) - \int_0^x K_{\alpha\beta}(x, y)(\sigma_1)_{\beta\gamma}\varphi_\gamma^{(0)}(E, y)\,\mathrm{d}y , \tag{2.4}$$

and the matrix elements of K are

$$K_{\alpha\beta}(x, y) = \mu \int_{-\infty}^{+\infty} \mathrm{d}\sigma(E)\varphi_\alpha(E, x)\varphi_\beta^{(0)}(E, y) . \tag{2.5}$$

Eq. (2.2) follows from the continuity equation

$$\mu(E - E')\int_0^x \varphi_\beta^{(0)}(E', y)(\sigma_1)_{\beta\gamma}\varphi_\gamma^{(0)}(E, y)\,\mathrm{d}y = \varphi^{(0)}(E, x)\,\frac{\partial\varphi^{(0)}(E', x)}{\partial x} - \varphi^{(0)}(E', x)\,\frac{\partial\varphi^{(0)}(E, x)}{\partial x} , \tag{2.6}$$

after substituting Eqs. (2.5) and (2.6) in Eq. (2.4).

We remark here that the form of the transformation given by Eq. (2.2) is very similar to that already established ([1]) in the case of non-relativistic

kinematics (cfr. Eq. (1.9) of the paper I), we have here a term $\cos(Q-Q^{(0)})$ instead of 1, as it must appear since the phase shifts approach a finite limit as $E\to\infty$ in the relativistic case. The moments

$$K_n(x)=\int_{-\infty}^{+\infty}\mathrm{d}\sigma(E)E^n\varphi(E,x)\varphi^{(0)}(E,x)\,,$$

$$H_n(x)=\int_{-\infty}^{+\infty}\mathrm{d}\sigma(E)E^n\varphi(E,x)\varphi^{(0)\prime}(E,x)\,,$$

needed for the asymptotic expansions, are of course different from the non relativistic ones and in particular they satisfy to slightly more complicated recurrence relations due to the new form of the equation of motion Eq. (2.1).

It is now straightforward to apply the same procedure followed in the non-relativistic case, although one is forced to more lengthy calculations.

One arrives at the result

$$\text{(2.7)}\quad \operatorname{tg}\delta_l=-\frac{k}{E}\operatorname{tg}\delta\lambda\left\{1+\frac{1}{E^2}\left[\mu^2\left(\frac{1}{2}+\frac{\delta\lambda}{\sin 2\,\delta\lambda}\right)+\frac{l(l+1)}{\sin 2\,\delta\lambda}\int_0^\infty\frac{(\delta V)'}{x}\,\mathrm{d}x\right]+O\left(\frac{1}{E^4}\right)\right\}.$$

If we compare this expansion with the corresponding one for a Dirac particle as given by Eq. (1.10) we find that they differ by a term

$$-\frac{k}{E^3}\operatorname{tg}\delta\lambda\,\frac{1}{\sin 2\,\delta\lambda}\,(l(l+1)-(j+\tfrac{1}{2})^2)\int_0^\infty\frac{(\delta V)'}{x}\,\mathrm{d}x\,,$$

which is entirely due to the spin-orbit coupling. Indeed $l(l+1)-(j+\frac{1}{2})^2$ coincides with the expectation values of the operator $\boldsymbol{L}\cdot\boldsymbol{\sigma}$ in the substates $j=l+\frac{1}{2}$ and $j=l-\frac{1}{2}$ and $(1/x)(\mathrm{d}/\mathrm{d}x)\,\delta V$ represents the familiar radial dependence of the Thomas precession.

This result can be also obtained iterating the Dirac equation and retaining terms up to the order $1/E^2$. It is preferable to work directly with the Dirac equation, because of its linearity in the potential. This yields considerable simplicity in the actual calculations.

4. – Final remarks.

The expansions (1.10) and (2.7) serve mostly to follow quantitatively at large energies how the phase shifts change in correspondence to a different shape of the interacting potential. Naturally they are useless for large impact

parameters. In this case, however, the incoming particle is moving outside the region where δV is appreciably different from zero and a perturbation expansion can instead serve the scope. It is probably also possible to obtain expansions valid for large angular momenta, following a procedure similar to that used in the case of non-relativistic kinematics [4].

For the total scattering amplitude at high energies, since no difference exists between the partial waves corresponding to the same j and opposite parities, the same considerations already given in the Schrödinger case will continue to be valid [5].

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(4) See Sect. **3** of the paper mentioned in I.

(5) See Sect. **4** of the same paper.

RIASSUNTO

Si ottengono sviluppi asintotici delle ampiezze d'urto per grandi valori dell'energia, in problemi di urto in un campo esterno a simmetria sferica, di particelle di Dirac e di Klein-Gordon. Viene stabilita una forma integrale per il generico coefficiente dello sviluppo. Essa ne permette la valutazione mediante formule ricorrenti.

FONDAZIONE FRANCESCO SOMAINI
PRESSO IL TEMPIO VOLTIANO, A COMO

BANDI DI CONCORSI
AL PREMIO E ALLA BORSA PER IL 1958

Con lo scopo di premiare e incoraggiare nel nome di ALESSANDRO VOLTA gli studi di Fisica in Italia, la « Fondazione Francesco Somaini », presso il Tempio Voltiano a Como, indice i seguenti concorsi.

A) **Concorso al "Premio Triennale per la Fisica Francesco Somaini" per il 1958.** di L. 1 500 000 (un milione e cinquecentomila) nette, da assegnarsi al concorrente che, fra quelli che la Commissione Giudicatrice giudicherà in senso assoluto meritevoli del premio per i risultati conseguiti nello studio della Fisica durante il Triennio 1° Luglio 1955-30 Giugno 1958, sia, a parere della Commissione stessa, il più meritevole.

B) **Concorso alla "Borsa Francesco Somaini per lo studio della Fisica" per il 1958.** di L. 750 000 (settecentocinquantamila) nette, da assegnarsi al concorrente che, tra quelli che la Commissione Giudicatrice giudicherà in senso assoluto meritevoli della Borsa, verrà dalla Commissione stessa giudicato il più meritevole, sia per titoli, preparazione scientifica, lavori già svolti e risultati già conseguiti nella Fisica, sia anche per il vantaggio che gli studi, per i quali è richiesta la Borsa, possono portare allo sviluppo della Fisica, in Italia.

1. – Ad entrambi i Concorsi possono prendere parte singolarmente i cittadini, d'ambo i sessi, italiani e svizzeri del Canton Ticino purchè di stirpe italiana. Sono esclusi dal Concorso i membri della Commissione Amministratrice e della Commissione Scientifica della « Fondazione Francesco Somaini ».

2. – Le norme particolareggiate dei singoli Concorsi verranno pubblicate in apposito volantino che potrà essere richiesto dagli interessati alla Segreteria della Fondazione presso il Tempio Voltiano a Como.

3. – La domanda, i documenti, i lavori, ecc., presentati dai singoli concorrenti dovranno pervenire, tra il 1° Gennaio e le ore 12 del 1° luglio 1958, alla Commissione Amministratrice della « Fondazione Francesco Somaini » a Como presso il Tempio Voltiano.

4. – Il Premio Triennale per la Fisica potrà essere anche conferito a uno studioso che non abbia preso parte al Concorso, ma sia stato segnalato da un Membro della Commissione Giudicatrice, con proposta motivata, come meritevole di particolare considerazione oppure ritenuto degno di premio dalla Commissione Giudicatrice, indipendentemente da ogni segnalazione.

* * *

La procedura dei suddetti Concorsi è regolata secondo lo Statuto della Fondazione, il quale è ostensibile a Como presso il Tempio Voltiano ed è depositato negli atti del Notaio Dr. Raoul Luzzani di Como.

Como, dal Tempio Voltiano, il giorno 6 *Agosto* 1955.

Il Segretario Conservatore del Tempio
CESARE MORLACCHI

Il Presidente Sindaco di Como
PAOLO PIADENI

[Il bando in esteso di detti Concorsi è stato pubblicato nel fascicolo di Gennaio del *Nuovo Cimento* 1956].

FASCICULES SPÉCIAUX

DU

SUPPLEMENTO

AUX VOLUMES DES SÉRIES IX ET X

DU

NUOVO CIMENTO

- **Comptes rendus de Colloques, Congrès, Conférences internationaux, organisés par la Société Italienne de Physique.**

  Colloque International de Mécanique Statistique (Florence, 1949).
  Congrès International sur les Rayons Cosmiques (Côme, 1949).
  Colloque International sur les Ultrasons (Rome, 1950).
  Colloque International d'Optique et Microondes (Milan, 1952).
  Congrès International sur les particules instables lourdes et sur les événements de haute énergie dans les Rayons Cosmiques (Padoue. 1954).
  Colloque International sur la Physique des solides et liquides (Varenna, 1954).
  Colloque International d'études sur l'Infrarouge (Parme, 1954).
  Conférence Internationale sur les particules élémentaires (Pise, 1955).
  Colloque International des études sur l'ionosphère (Venise, 1955).
  Colloque International sur les constantes fondamentales dela Physique (Turin,1956).
  Colloque International sur les rayons cosmiques (Varenna, 1957)*.
  Colloque International sur l'état condensé de sistèmes simples (Varenna, 1957)*.
  Colloque International sur les mésons et les particules récemment découvertes (Padoue-Venise, 1957) (*s'adresser à l'* Istituto di fisica dell'Università - Padova - via Marzolo, 8).

  * sous presse

- **Cours tenus à Varenna à l'école internationale de Physique de la Société Italienne de Physique.**

  Révélation des particuls élémentaires, avec référence spéciale à la Radiation cosmique (Varenna, 1953).
  Physique des particules élémentaires et machines accéleratrices (Varenna, 1954).
  Questions de Physique Nucléaire: structure et modèles du noyau, nouvelles espèces d'atomes (positronium, atomes mésiques), Optique neutronique, moments nucléaires, processus physiques du réacteur nucléaire (Varenna, 1955).
  Propriétés magnétiques de la matière (Varenna, 1956).
  Physique de l'état solide (Varenna, 1957)*.

  * sous presse

- **Rapports sur des questions diverses de Physique.**

  Premier, Deuxième, Troisième, Quatrième, Cinquième Rapport sur les travaux de Physique en Russie (1953, 1956).
  Morpurgo et Franzinetti. « *An Introduction to the Physics of the new particles* » (1957).

Photoproduction of π^--Mesons Near Threshold and the Value of the Panofsky Ratio (*).

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(ricevuto il 24 Febbraio 1958)

Summary. — An examination of the cross-sections for the reactions

$$\gamma + d \to \begin{cases} 2n + \pi^+ \\ 2p + \pi^- \end{cases}$$

near threshold was performed. The comparison of the results of this examination with experiment is presented, and it is shown that the negative to positive ratio is constant in the photon energy range $(153 \div 193)$ MeV and equal to 1.4 (to an accuracy of 10%) in agreement with field theoretic results. The disagreement of these results with the values of the Panofsky ratio and the S-wave scattering phase shifts is discussed from the point of view of the two-component π^0-meson field hypothesis. A direct experimental test of this hypothesis is proposed.

1. – Introduction.

The interaction of low-energy π-mesons with nucleons is one of the most interesting problems in pion physics. The understanding of the interaction in the S-state leaves much room for improvement. The experimental data on this interaction are discussed in numerous papers and conference reports.

(*) A part of this work was performed in the Physical Institute of the USSR Academy of Sciences and reported in the *Padua-Venice Conference* (September 1957). The other part of the work was performed in the Department of Mathematical Physics, University of Birmingham where the author is at present a visitor.

At the last conference in Rochester it was stated [1,2] that the different experimental data on this interaction (the S-wave scattering phase shifts, the Panofsky ratio and the cross-sections for the photoproduction of π^- and π^+ mesons near threshold) are consistent between themselves within the limits of experimental error but inconsistent with the predictions from the threshold theorems and the dispersion relation approach [3]. The threshold value of the negative to positive pion ratio in photoproduction σ^-/σ^+ given by the field theoretical approach is equal to 1.4 if one uses for the renormalized coupling constant the commonly accepted value $f^2 = 0.08$ (*). From the experimental data on the Panofsky ratio, S-wave scattering phase shifts and σ^+ [4] one can get for σ^-/σ^+ a value of about 2.

Direct measurement of σ^-/σ^+ with extrapolation to the threshold [4] gives 1.8 The discussion of this puzzle is the purpose of this paper. The direct measurements of the value of σ^-/σ^+ near threshold are not free from objection. These measurements involve the detection of the pions of a given energy in the reactions:

$$\gamma + \mathrm{d} \to \begin{cases} 2\mathrm{n} + \pi^+ \,, & (1) \\ 2\mathrm{p} + \pi^- \,. & (2) \end{cases}$$

The measurement of low-energy pions is not a measurement of the threshold cross-sections σ^- and σ^+ because of the three body problem in the final state and the spread of the bremsstrahlung spectrum. One must have more detailed information on the reaction (2) before coming to conclusion about σ^- for the free nucleon. Then it is necessary to analyse in detail the mechanism of photoproduction of pions from deuterons taking into account all effects which contribute more than 10% (the accuracy of σ^+ measurement). The theory which one can get as a result of such a discussion must be compared in detail with experiment because it is at present impossible to treat the problem without making assumptions.

In Sect. **2** we discuss the theory of reactions (1) and (2) near threshold. In Sect. **3** we compare this theory with the detailed experimental data [5]

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(1) J. M. Cassels: *Proc. VII Ann. Rochester Conference* (1957).

(2) G. F. Chew: *Proc. VII Ann. Rochester Conference* (1957).

(3) Cfr. G. F. Chew, M. L. Goldberger, F. E. Low and Y. Nambu: *Phys. Rev.*, **106**, 1345 (1957).

(4) M. Beneventano, G. Bernardini, D. Carlson-Lee, G. Stoppini and L. Tau: *Nuovo Cimento*, **4**, 323 (1956).

(*) Strictly speaking the ratio $\sigma^{(-)}/\sigma^{(+)}$ does not depend on f^2. However according to (3) it depends on unknown parameter $N^{(-)}$. If one obtains $N^{(-)}$ from comparison of $\sigma^{(+)}$ with the experimental value of it, assuming $f^2 = 0.08$, the value of the ratio $\sigma^{(-)}/\sigma^+$ is 1.4.

on reaction (2). As a result of this comparison we give the value and energy dependence of the squared modulus of the amplitude of the reaction $\gamma + n \rightarrow \rightarrow p + \pi^-$ (Fig. 5). Our results are in agreement with the field theoretical approach and in disagreement with the experimental data on the S-wave phase shift and the Panofsky ratio. In Sect. **4** we briefly discuss the new situation in the pion-nucleon S-wave interaction (*).

2. – Theory.

The theory of the photoproduction of mesons from deuterons in the impulse approximation was discussed in several papers [6,7]. Our task is to investigate carefully the effect of the final state interaction in reactions (1) and (2). Some aspects of this problem we have discussed in reference [7].

The total Hamiltonian of the system has the form:

$$H = H_0 + H' + V + V_c\,, \tag{3}$$

H_0 is the sum of the Hamiltonians for the free meson field and the interacting nucleons, H' contains the terms in the Hamiltonian involving the interaction between the nucleons. The meson field and electromagnetic field V and V_c are nuclear and Coulomb potentials of the meson in the presence of the nucleons. The transition operator for the problem can be written in the form:

$$T = \left[1 + (V + V_c)\,\frac{1}{E - H_0 + i\varepsilon}\right] H'\,, \tag{4}$$

if one treats the last three terms of (3) as a perturbation. The first term of (4) is the usual impulse approximation. The operator H' has the form:

$$\sum_\nu \exp\left[i(\boldsymbol{\varkappa} - \boldsymbol{k})\boldsymbol{\tau}_\nu\right]\left[(\boldsymbol{\sigma}\boldsymbol{K}^\nu) + L^\nu\right]\,, \tag{5}$$

here the $\boldsymbol{\sigma}^\nu$ are the Pauli matrices, $\boldsymbol{\varkappa}$ and $\boldsymbol{k}$ are the momentum vectors of photon and meson respectively, $\boldsymbol{K}^\nu$ and L^ν are the « spin flip » and « non-spin flip »

[5] M. I. Adamovich, V. I. Veksler, G. V. Kusmicheva, V. G. Larionova and S. Charlamov: *Proc. CERN Symposium* (Geneva, 1956), and private communication. The author is indebted to this group of experimentalists and especially to M. I. Adamovich for detailed information about their results.

(*) A more complete discussion will be given in a separate paper.

[6] G. F. Chew and N. W. Lewis: *Phys. Rev.*, **84**, 779 (1951).

[7] A. M. Baldin: *Thesis* (Physical Institute of the USSR Academy of Sciences, 1953); *CERN Symposium Proc.*, **2**, 272 (1956).

parts of the transition operator for free nucleons. A simple calculation [7] results in the following expression for the cross-section of the reactions (1) and (2) (*)

$$\text{(6)}\qquad \mathrm{d}\sigma = \frac{16}{(2\pi)^2}\left\{|K|^2\left[|I_a|^2\cdot\frac{2}{3}+\frac{1}{3}|I_s|^2\right]+|L|^2|I_a|^2\right\}'\delta(E_0-E_f)\,\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q}\,,$$

where $\boldsymbol{p}=(\boldsymbol{p}_1-\boldsymbol{p}_2)/2$ is the relative momentum of the two nucleons and $\boldsymbol{q}=(\boldsymbol{p}_1+\boldsymbol{p}_2)/2$

$$I_a=\int\varphi_{fa}\exp[i\boldsymbol{qr}]\varphi_d\,\mathrm{d}\boldsymbol{r}\qquad\text{and}\qquad I_s=\int\varphi_{fs}\exp[i\boldsymbol{qr}]\varphi_d\,\mathrm{d}\boldsymbol{r}\,,$$

$$\varphi_\alpha=\sqrt{\frac{7\alpha}{9\pi}}\,\frac{\exp[-\alpha r]-\exp[-7\alpha r]}{r}\quad\text{deuteron wave function } (^+)\,.$$

The suffix « a » or « s » means that the co-ordinate part of the wave function for the relative motion of the nucleons is antisymmetrical or symmetrical respectively φ_{fa} and φ_{fs} were taken to be:

$$\text{(7)}\qquad \varphi_{fa}=\frac{1}{2}\,\frac{1}{(2\pi)^{\frac{3}{2}}}\left(\exp[i\boldsymbol{pr}]-\exp[-i\boldsymbol{pr}]\right),$$

$$\text{(8)}\qquad \varphi_{fs}=\frac{1}{2}\,\frac{1}{(2\pi)^{\frac{3}{2}}}\left[\exp[i\boldsymbol{pr}]+\exp[-i\boldsymbol{pr}]+\frac{1-\exp[-2i\delta]}{ip}\,\frac{\exp[-ipr]}{r}\right],$$

for reaction (1). Here δ is the phase shift for neutron-neutron S-wave scattering. This last wave function substantially exceeds the true wave function at small distance r. The contribution from small r involves two other effects which are difficult to evaluate at present. The first is the competing process $\gamma+\mathrm{d}\to\mathrm{p}+\mathrm{n}$ and the second is nuclear multiple scattering of the meson. Multiple scattering corrections give contributions of the order of $\sim a/r$, where a is the scattering length. For non-relativistic mesons $a/r \lesssim 0.1$ if $r \gtrsim 1$. Both of the above mentioned effects decrease the wave function at small distances.

To take into account all these corrections let us assume that $\varphi_{fa,s}$ equals zero for $r \lesssim 1$. Such an assumption is also suggested by the $A^{\frac{2}{3}}$ dependence

(*) Here we have neglected recoil terms in K and L depending on the velocity of the nucleons [7]. They can give a contribution to our results of about 10%. But if one neglects such terms in discussing the $\pi^\pm$ output from a deuterium target an error of about 20% can be involved. (This term can give contributions of opposite sign to the π^+ and π^- amplitudes. The terms from the second part of (4) have also been omitted. For a discussion of these terms see below).

($^+$) $\hbar=c=\mu=1$ where μ is the meson mass.

of photoproduction of mesons from nuclei. We therefore subtract from $I_{a,s}$ the values

$$\delta I_{a,s} = \int_0^1 r^2 \mathrm{d}r \int \varphi_{a,s} \exp\,[i\boldsymbol{q}\boldsymbol{r}]\varphi_d \mathrm{d}\Omega_r\,.$$

To be on the safe side we should consider only those values of our variables p and q for which this correction to the cross-section is less than 10%. However, as we shall see later, this assumption gives rather good agreement with a pronounced experimental feature even in those regions of p and q where this correction amounts to about 25%.

Evaluation of the integrals involved results in the following expressions for the cross-section of reaction (1)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}p\,\mathrm{d}q} = A(p,q)\,|K|^2 + B(p,q)\,|L|^2\,, \tag{9}$$

where

$$A(p,q) = \frac{3\alpha}{\pi^2}\,p\left(1-\frac{p^2+q^2}{M\varkappa}\right)\left\{\frac{4pq}{(\alpha^2+p^2+q^2)^2-4p^2q^2} - \frac{1}{3}\,\frac{1}{\alpha^2+p^2+q^2}\ln\frac{\alpha^2+(p+q)^2}{\alpha^2+(p-q)^2} + \right.$$
$$+\frac{2}{3}\,\frac{\sin^2\delta}{pq}\left[\left(\operatorname{arctg}\frac{2q\alpha}{\alpha^2+p^2-q^2} - 1.8q - \frac{pq\operatorname{ctg}\delta}{2.1}\right)^2 - \frac{1}{4}\left(\ln\frac{\alpha^2+(p+q)^2}{\alpha^2+(p-q)^2}\right)^2\right] +$$
$$\left. +\frac{1}{3}\,\frac{\sin 2\delta}{pq}\left(\operatorname{arctg}\frac{2q\alpha}{\alpha^2+p^2-q^2} - 1.8q - \frac{pq\operatorname{ctg}\delta}{2.1}\right)\ln\frac{\alpha^2+(p+q)^2}{\alpha^2+(p-q)^2}\right\},$$

$$B(p,q) = \frac{3\alpha}{\pi^2}\,p\left(1-\frac{p^2+q^2}{M\varkappa}\right)\left\{\frac{4pq}{(\alpha^2+p^2+q^2)^2-4p^2q^2} - \right.$$
$$\left. - \frac{1}{\alpha^2+p^2+q^2}\ln\frac{\alpha^2+(p+q)^2}{\alpha^2+(p-q)^2}\right\},$$

A and B are practically independent of $\varkappa$. The region of the variables consistent with the conservation laws is

$$0 \leqslant p \leqslant \sqrt{M\left[q_0 - \frac{1}{2\varkappa} - \varepsilon - \frac{1}{q_0}\,(q-q_0)^2\right]}\,,$$

here $q_0 = \varkappa/(2+(\varkappa/M))$, M is the neutron mass ε includes the binding energy of the deuteron and the neutron proton mass difference. $A(p,q)$ and $B(p,q)$ are peaked at $p=q$. This peaking is connected with the weak binding of

the deuteron. For free nucleons the dependence of the transition amplitude on p and q would contain a factor $\delta(\boldsymbol{p} \pm \boldsymbol{q})$.

In the region $p < 0.2$ $A(p, q) \gg B(p, q)$ because $A(p, q)|K|^2$ is the spin flipping part of the cross-section and involves enhancement due to the final state interaction in the ${}^1S_0$-state of the neutrons.

The correction $\delta I_{a.s}$ gives contributions $\lesssim 10\%$ in the region $p \gtrsim q$ and about 25% in the region of small p. These corrections affect the nucleon S-wave contribution and have very little effect on other partial waves which give the main contribution at $p \gtrsim q$ and a small contribution for small p.

For the discussion of reaction (2) it is necessary to take into account the Coulomb forces between all three particles. Roughly speaking the contribution of the Coulomb correction is of the order of magnitude of $\sim \pi(e^2/V)$ (V, relative velocity of interacting particles). The protons, being heavy particles, have much smaller velocities than mesons and must be more strongly influenced by the Coulomb interaction. But at the same time the nucleons have shorter wave lenghts and for them the estimate $\pi(e^2/V)$ is applicable only in the region of very small p. It is necessary also to keep in mind that the region of the small relative velocities (small p) is the region of enhancement due to the nuclear final state interaction. It is clear from this brief discussion that for the evaluation of the role of the Coulomb repulsion detailed calculations are necessary.

Instead of the functions (7) and (8) we take the exact Coulomb wave function:

$$\text{(10)} \qquad \frac{1}{(2\pi)^{\frac{3}{2}}} \exp[i\boldsymbol{pr}] \to \frac{1}{(2\pi)^{\frac{3}{2}}} \exp\left[-\frac{\pi}{2}\eta\right] \cdot \\ \cdot |\Gamma(1+i\eta)| \exp[i\boldsymbol{pr}] F[i\eta, 1, -i(pr + \boldsymbol{pr})] ,$$

$\eta = (e^2/V)$, F, hypergeometrical function

$$\text{(11)} \qquad \frac{1 - \exp[-2i\delta]}{ip} \frac{\exp[-ipr]}{r} \to 2\exp[-i\delta_0](1 - \exp[-2i\delta_c]) \cdot \\ \cdot \exp\left[-\frac{\pi}{2}\eta\right] \exp[-ipr] W_1(-i\eta + 1, 2, 2ipr) ,$$

where W_1, Whittaker function, $\delta_0 = \arg \Gamma(1+i\eta)$; $\delta_c - S$ phase-shift in p-p scattering.

The method of evaluation of the integrals:

$$\text{(12)} \qquad J_0 = \int \exp[i\boldsymbol{nr}] \frac{\exp[-\alpha r]}{r} F[-i\eta, 1, i(pr + \boldsymbol{pr})] \mathrm{d}\boldsymbol{r} ,$$

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can be found in reference ([8]). Expressing F as a contour integral (see Fig. 1)

$$(13)\qquad F(-i\eta, 1, \beta) = \frac{1}{2\pi i}\exp[i\beta]\oint_{\gamma}\left(1-\frac{1}{x}\right)^{-i\eta-1}\frac{\mathrm{d}x}{x}\cdot\exp[i\beta x]\,,$$

and changing the order of the ordinary and the contour integration one can get after calculation

$$(14)\qquad J_0 = \frac{4\pi[\alpha^2 - p^2 + q^2 - 2i\alpha p]^{i\eta}}{[\alpha^2 + p^2 + q^2 + 2pq\cos\zeta]^{i\eta+1}}\,.$$

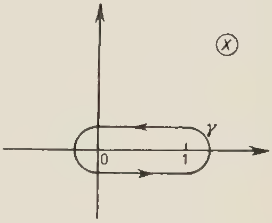

Fig. 1.

The integral with Whittaker function cannot be evaluated in terms of elementary functions. Expressing W as a contour integral (see Fig. 2)

$$(15)\qquad W_1(i\eta+1, 2, z) = \frac{1}{2\pi i}\oint_{\gamma_1}\left(1-\frac{1}{x}\right)^{-i\eta-1}\frac{\exp[xz]}{z}\,\frac{\mathrm{d}x}{x^2}\,,$$

one can get after performing simple transformations

$$(16)\qquad \int \exp[ipr]W_1(i\eta+1, 2, -2ipr)\exp[i\boldsymbol{nr}] = \frac{\pi}{pq}\,\frac{1-\exp[2\pi\eta]}{2\pi\eta}\,\cdot$$

$$\cdot\int_0^1 \exp[i\eta\ln t]\left\{\frac{1}{\dfrac{p-q+i\alpha}{p+q-i\alpha}+t} - \frac{1}{\dfrac{p+q+i\alpha}{p-q-i\alpha}+t}\right\}\mathrm{d}t = \frac{\pi}{pq}\cdot\frac{1}{C^2}(f_1+if_2)\,,$$

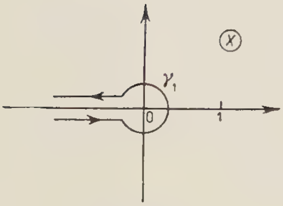

Fig. 2.

$$C^2 = \frac{2\pi\eta}{\exp[2\pi\eta]-1}\,,$$

f_1 and f_2 are real functions which can be evaluated numerically. For small η one can use the expansion

$$f_1 \simeq f_1^{(0)} + \eta f_1^{(1)}\,,\qquad f_2 \simeq f_2^{(0)} + \eta f_2^{(1)}\,,$$

$f_1^{(1)}$ and $f_2^{(1)}$ can be evaluated with rather low accuracy. Expanding I_0 and taking into account only the first term in η permits simple calculation of integrals:

$$\iint \frac{\mathrm{d}\sigma}{\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q}}\,\mathrm{d}\Omega_p\,\mathrm{d}\Omega_q\,.$$

([8]) A. SOMMERFELD: *Atombau*, II (Braunschweig, 1956).

The results of these calculations are

$$\frac{\mathrm{d}\sigma}{\mathrm{d}p\,\mathrm{d}q} = A^c(p,q)\,|K^{(-)}|^2 + B^c(p,q)\,|L^{(-)}|^2\,, \tag{17}$$

where (*)

$$\begin{aligned} A^c(p,q) = {} & \frac{3\alpha}{\pi^2}\,p\left(1-\frac{p^2+q^2}{M\varkappa}\right)C^2\left\{\left(1+2\eta\,\mathrm{arctg}\,\frac{2\alpha p}{\alpha^2+q^2-p^2}\right)\cdot\right. \\ & \cdot\left[\frac{4pq}{(\alpha^2+p^2+q^2)^2-4p^2q^2}-\frac{1}{3}\,\frac{1}{\alpha^2+p^2+q^2}\ln\frac{\alpha^2+(p+q)^2}{\alpha^2+(p-q)^2}\right]+ \\ & +\frac{2}{3}\,\frac{\sin^2\delta_c}{C^4pq}\,[f_1^2+f_2'^2]-\frac{2}{3}\,\frac{\sin\delta_c}{pq}\cdot\left(1+\eta\,\mathrm{arctg}\,\frac{2\alpha p}{\alpha^2+q^2-p^2}\right)\cdot \\ & \cdot\,[f_2'\cos(\delta_c+\delta_0)+f_1\sin(\delta_c+\delta_0)]\ln\frac{\alpha^2+(p+q)^2}{\alpha^2+(p-q)^2}\,, \end{aligned} \tag{18a}$$

$$B^c(p,q) = \left(1-2\eta\,\mathrm{arctg}\,\frac{\alpha^2+q^2-p^2}{2p\alpha}\right)B(p,q)\,, \tag{18b}$$

and

$$f_2' = f_2 + q\left[1.8 + \frac{pC^2\,\mathrm{ctg}\,\delta_c}{2.1}\right].$$

In the region of $p \lesssim 0.1$, $A^c \lesssim 3A$ and the Coulomb corrections are very important. The large difference between B^c and B is not essential because B^c and B are very small in this region. In the region of $0.2 < p \lesssim 0.4$, A^c and B^c are smaller respectively than A and B by about 10%. At $p > 0.4$ the difference between A^c and A and B^c and B is beyond the accuracy of our calculations. The numerical values for A, A^c are given in tables I and II for the most essential region of our variables.

TABLE I. – $A(p, q)$.

p \ q	0.2	0.4	0.6	0.8
0.2	0.18	0.25	0.20	0.14
0.4	0.13	0.30	0.26	0.16
0.6		0.20	0.38	0.27
0.8		0.09	0.30	0.49
1		0.08	0.20	

(*) We do not expand C^2 here because one can show that eqs. (18) are applicable in the region of small p. At $p \lesssim 0.1$, $f_2 \approx -\pi$ and $f_1 \approx 0$.

TABLE II (*). – $A^c(p, q)$.

$p \backslash q$	0.2	0.4	0.6	0.8
0.2	0.16	0.22	0.17	0.12
0.4	0.12	0.28	0.23	0.14
0.6		0.20	0.38	0.27
0.8		0.09	0.30	0.49
1		0.05	0.15	0.37

(*) A^c is given without the correction from equation (24) because this correction is dependent on $\varkappa$ and it is more convenient to take into account the factor (24) separately.

To take into account the nucleon-meson Coulomb correction (*) it is necessary, according to (4), to multiply the transition matrix $\langle \boldsymbol{p}', \boldsymbol{q}', \boldsymbol{k} j m_j | T | i\rangle$ by the matrix

$$\langle \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{k}, j, m \,|\, (V_1^c + V_2^c) \frac{1}{E - H_0 + i\varepsilon} \,|\boldsymbol{p}'\boldsymbol{q}'\boldsymbol{k}'j'm_j'\rangle\,, \tag{19}$$

where $V_1^c = e^2/|\boldsymbol{r}_1 - \boldsymbol{r}_3|$, $V_2^c = e^2/|\boldsymbol{r}_2 - \boldsymbol{r}_3|$.

$\boldsymbol{r}_1$, $\boldsymbol{r}_2$, $\boldsymbol{r}_3$, are the radius vectors of the protons and the meson respectively, j and m_j the spin of the nucleon system and its projection. Straightforward calculations give for (19) (+)

$$\delta_{jj'}\,\delta_{m_j m_j'}\,\delta(-2\boldsymbol{q} + 2\boldsymbol{q}' - \boldsymbol{k} + \boldsymbol{k}')\,\frac{4\pi e^2\cdot 2}{(2\pi)^3[|\boldsymbol{k} - \boldsymbol{k}'|^2 + \beta^2]}\cdot \int \varphi_{pj}^*(\boldsymbol{r}) \cos\left(\boldsymbol{r}, \frac{\boldsymbol{k} - \boldsymbol{k}'}{2}\right)\varphi_{p'j'}\,\mathrm{d}\boldsymbol{r}\,. \tag{20}$$

Denoting the above integral by $I_{pp'}^j(\boldsymbol{k} - \boldsymbol{k}')$ and introducing new variables, one can obtain the following equation for the correction of the transition matrix:

$$\frac{e^2}{\pi}\int I_{pp'}^j\left(\frac{\boldsymbol{\gamma} - \boldsymbol{t}}{2}\right)\frac{\mathrm{d}\boldsymbol{p}'\,\mathrm{d}\boldsymbol{t}}{\dfrac{p^2 - p'^2}{M} + \dfrac{\gamma^2 - t^2}{2} + i\varepsilon}\cdot\frac{1}{|\boldsymbol{\gamma} - \boldsymbol{t}|^2 + \beta^2}\cdot \left\langle \boldsymbol{p}', \boldsymbol{q} + \frac{\boldsymbol{\gamma} - \boldsymbol{t}}{2}, \boldsymbol{k} - (\boldsymbol{\gamma} - \boldsymbol{t})\,|\,T\,|\,i\right\rangle, \tag{21}$$

$$\boldsymbol{\gamma} = \boldsymbol{k} - \frac{1}{1 + 2M}(\boldsymbol{k} + 2\boldsymbol{q})\,.$$

(*) This question was also investigated by M. MORAVCSIK. The author is indebted to Dr. MORAVCSIK for informing him about this work and sending preprints of other works on photoproduction.

(+) Here β is the real physical cut-off taking into consideration inner bremsstrahlung.

Equation (21) can give a contribution greater than 10% only for meson momentum less than ~ 0.3. A small momentum of the meson corresponds to large p (according to the conservation laws) and, as was shown above, small contribution of the nucleon S-wave part. So one can neglect the S-wave part of φ_{pj} and $\varphi_{p'j'}$, and I_{pp} reduces to a combination of the δ-functions.

Performing the integrations on the assumption that the integrand has no singularities except those written explicitly, one can obtain

$$(22)\qquad -ie^2\left\{\frac{1}{|\boldsymbol{k}-(\boldsymbol{q}+\boldsymbol{p})/M|}\ln\left(1+2i\frac{|\boldsymbol{k}-(\boldsymbol{q}+\boldsymbol{p})/M|}{\beta}\right)+\right.$$
$$\left.+\frac{1}{|\boldsymbol{k}-(\boldsymbol{q}-\boldsymbol{p})/M|}\ln\left(1+2i\frac{|\boldsymbol{k}-(\boldsymbol{q}-\boldsymbol{p})/M|}{\beta}\right)\right\}\langle\boldsymbol{p},\boldsymbol{q},\boldsymbol{k}|T|i\rangle\,.$$

For $k \lesssim 0.3$ it is possible to neglect the interaction between the nucleons. This means that $\langle\boldsymbol{p},\boldsymbol{q},\boldsymbol{k}|T|i\rangle$ is a real quantity and it is necessary to take into account only the real part of the correction. For $\beta\to 0$ we have:

$$\frac{\pi e^2}{2}\left(\frac{1}{|\boldsymbol{k}-(\boldsymbol{q}+\boldsymbol{p})/M|}+\frac{1}{|\boldsymbol{k}-(\boldsymbol{q}-\boldsymbol{p})/M|}\right)\langle\boldsymbol{p},\boldsymbol{q},\boldsymbol{k}|T|i\rangle\,.$$

It is easy to show that

$$(23)\qquad \left(\frac{p/M}{|\boldsymbol{k}-\boldsymbol{q}/M|}\right)^2\ll 1\,,$$

almost in the whole region of interest.

Neglecting $p^2/(M^2|k-q/M|^2)$ we can obtain that the correction on meson-proton Coulomb interaction to the cross-section of reaction (2) reduces to the factor:

$$(24)\qquad \left(1+\frac{2\pi e^2}{|\boldsymbol{k}-\boldsymbol{q}/M|}\right)\simeq\left(1+\frac{2\pi e^2}{|\boldsymbol{k}-\boldsymbol{\varkappa}/2M|}\right)=$$
$$=\left(1+\frac{0.046}{\sqrt{(\varkappa^2-1.09)-0.32(\varkappa-1)q^2-0.32\varkappa p^2}}\right).$$

Our final result is practically equivalent to the operation of multiplying the cross-section by $|\Psi(0)|^2$, where Ψ is the wave function of the meson in the Coulomb field of the protons. This means that it is valid only when the wave lengths of meson-nucleons relative motions are large in comparison with the size of the system:

$$\frac{1}{\lambda}\approx\left|\boldsymbol{k}-\frac{\boldsymbol{\varkappa}}{2M}\right|<\alpha=0.31\,.$$

For $\lambda\!\!\!\bar{} \sim 1/\alpha$ the correction is about 13%. For $1/\lambda\!\!\!\bar{} > \alpha$ the correction (24) is not valid, but the real correction is beyond the accuracy of our calculations and must be put equal to zero. The last statement follows from a simple analysis of equation (18*b*) which describes a very similar problem.

The restriction (23) for the applicability of (24) practically coincides with the criterion of applicability of perturbation theory because in the region of interest $p \lesssim 1$.

3. – Comparison with experiment (*).

For the most part our discussion deals with experimental results obtained in the Physical Institute of the U.S.S.R. Academy of Sciences ([5]). This experiment gives complete information about the process (2) for energies of the photons from the threshold up to ~ 190 MeV because all three final particles are detected. The coefficient B in the region $p < 0.3$ is negligible. This permits us to write

$$\frac{\displaystyle\int \mathrm{d}q \int_0^{0.3} \frac{\mathrm{d}\sigma}{\mathrm{d}p\,\mathrm{d}q}\,\mathrm{d}p}{\displaystyle\int \mathrm{d}q \int \frac{\mathrm{d}\sigma}{\mathrm{d}p\,\mathrm{d}q}} = \frac{\displaystyle\int \mathrm{d}q \int_0^{0.3} A(p, q)\,\mathrm{d}p}{\displaystyle\iint A(p, q)\,\mathrm{d}p\,\mathrm{d}q + \frac{|\bar{L}|^2}{|\bar{K}|^2}\iint B(p, q)\,\mathrm{d}p\,\mathrm{d}q}\,.$$

Comparison of this value with experiment gives $|\bar{L}|^2/|\bar{K}|^2 \lesssim 0.1$. This supports the well known result that the S-state of meson and nucleon gives the main contribution to the cross-section of photoproduction of pions near threshold. Within the limits of our accuracy $|\bar{L}|^2 \sim 0$.

To check our theory the following characteristics of the process were obtained

1) $$\int \frac{\mathrm{d}\sigma}{\mathrm{d}p\,\mathrm{d}q}\,\mathrm{d}q = \varphi_1(T) \qquad \text{where} \qquad T = \frac{p^2}{M}\,,$$

and

2) $$\int \frac{\mathrm{d}\sigma}{\mathrm{d}p\,\mathrm{d}\varepsilon}\,\mathrm{d}p = \varphi_2(\varepsilon) \qquad \text{where} \qquad \varepsilon = (p - q)\,.$$

These characteristics are very sensitive to the description of the relative motion

(*) The detailed discussion of the experimental data will be published in the *Journ. Exp. Theor. Phys. USSR.*

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of the protons. Experimental data for $\varphi_1(T)$ and $\varphi_2(\varepsilon)$ are shown in Fig. 3 and Fig. 4. The solid curves are the results of numerical integration in (26) and (27) with $\mathrm{d}\sigma/\mathrm{d}p\,\mathrm{d}q$ from (17), (18) and (23) and with $K^-|^2=0.8\cdot10^{-27}$ cm².

The last value corresponds to the field theoretical prediction with the coupling constant $f^2 = 0.08$.

As one can see the agreement with experiment is very good. This permits us to discuss the dependence of $|K^-|^2$ on $\varkappa$.

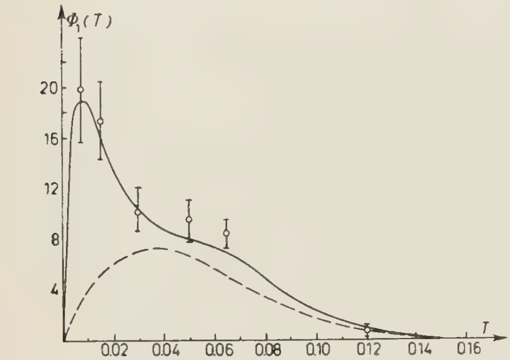

Fig. 3.

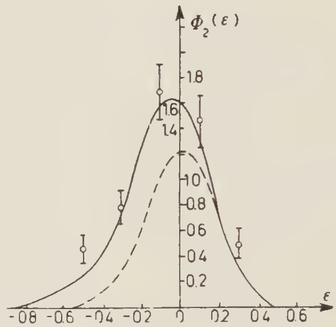

Fig. 4.

Experimental data for the total cross-section of reaction (2) are:

$\varkappa$ (in MeV)	156.5	163.5	170	181
$\sigma\cdot10^{29}$ cm^{-2}	4.6 ± 0.6	6.4 ± 0.8	9.0 ± 1.0	11 ± 1.7

Dividing σ by the results of the numerical integration $\iint A^c(p, q)\,\mathrm{d}p\,\mathrm{d}q$ one can obtain the dependence of $|K^{(-)}|^2$ on σ presented in Fig. 5. In Fig. 5 are also given data about $|K^{(+)}|^2$ from the important work of BENEVENTANO *et al.* (4). It is necessary to stress that our results are not in disagreement with experimental data on σ^-/σ^+ of reference (4), but only with the interpretation of these data. In the work (4) the measurement of the σ^-/σ^+ ratio was performed by counting only π^- and π^+ mesons of a given energy. As was shown above, this ratio is dependent on the energy of the meson. Coulomb interactions are responsible for increasing σ^- with decreasing meson energy. For high meson energy the positive Coulomb correction (meson-proton interaction) is negligible but the negative Coulomb correction (p-p interaction)

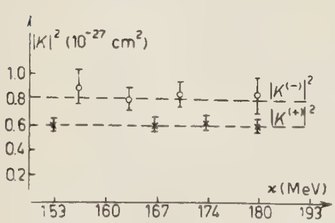

Fig. 5.

can give an appreciable contribution. For low meson energy ($\lesssim$ 10 MeV) the negative Coulomb correction is negligible but the positive correction can give a contribution of more than 10%. In the work of BENEVENTANO *et al* (4) such behaviour of σ^-/σ^+ was extrapolated to very low meson energy and it was assumed that the resulting value 1.87 is the threshold ratio σ^-/σ^+ for free nucleons (*). This result is in disagreement with our's, but the highest value of σ^-/σ^+ which was actually observed in the experiment of BENEVENTANO *et al.* was 1.5 at a meson energy of about $(10\div15)$ MeV. For such meson energies the positive Coulomb correction is, according to (24), about 10% and the negative correction, according to Table II and the energy conservation law, is negligible. Introducing this correction, we obtain excellent agreement with our result. Similar results can be obtained from the values of σ^-/σ^+ found in reference (4) and attributed to another « effective γ-ray energy », but, as was pointed out in Sect. **1**, such an analysis is not very conclusive. For the function $\varphi_1(T)$ and $\varphi_2(\varepsilon)$ the contribution of the positive Coulomb correction is very small.

4. – Discussion.

The situation can be summarized as follows: The analysis of experiments involves careful measurement of reaction (2) very near the threshold and shows that the σ^-/σ^+ ratio for free nucleons is independent of the energy of the photon and equals 1.4 ($\sim$10% accuracy). This behaviour and the value of the σ^-/σ^+ ratio is in very close agreement with the predictions of field theory with a coupling constant of $f^2 = 0.08$.

These conclusions contradict only the *interpretation* of the previous experimental data on direct measurements of the σ^-/σ^+ ratio but not the experimental results themselves. The disagreement of our results with the data on the Panofsky ratio and S-wave phase shifts in pion-nucleon scattering is, somewhat of a puzzle. From the σ^- obtained in the present work and from the well known Orear data on $|\delta_1 - \delta_3|$ one can obtain a Panofsky ratio of 2.4 ± 0.4 against 1.5 ± 0.1 as reported by CASSELS (1). If this disagreement is not a result of a trivial overestimation of the experimental accuracy, then it is necessary to look for a more speculative explanation.

Let us suppose that there exists (apart from the commonly accepted π^0-meson) a π_0^0-meson which has isotopic spin zero. The first consequence of the existence of such a meson is that it is necessary to perform a new analysis

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(*) Some objection against the procedure of extrapolation used by BENEVENTANO *et al.* was also discussed by M. MORAVCSIK (preprint).

of experiments on π^--proton interactions. It is then clear that the value of δ_1 can be changed.

The second consequence is that if one assumes the mass of ordinary π^0 to be approximately equal to the mass of the $\pi^\pm$-mesons, then the Panofsky ratio involves mainly the π_0^0 meson and the observed mass 264 is the mass of π_0^0. This assumption may be made plausible on the basis of the isotopic invariance: The values of $(m^\pm - m^0)/m^\pm$ (where $m^\pm$ and m^0 are the masses of the charged and neutral particles involved in the isotopic multiplets) are: for nucleons 0.15 % and for Σ-particles 0.3 %. But for π-mesons it is 3.3 %, i.e. about 10 times larger. The last value is closer to the ratio $(M_\Sigma - M_\Lambda)/M_\Sigma = 6.5\%$ (where M_Σ and M_Λ are the masses of the Σ and Λ particles) than to the first two.

The π^0-meson properties obtained from absorption of mesons in hydrogen and deuterium are now, according to our hypothesis, the properties of the π_0^0-meson, i.e. π_0^0 is pseudoscalar and strongly coupled with nucleons.

Then we see that the existence of π_0^0 would completely change the very much discussed connection between important low energy parameters in pion physics. I do not know of any experimental result which would be in direct contradiction with the assumption about the existence of π_0^0. Our knowledge of the transformation properties of the π-mesic field is based on numerous tests of the consequences of the isotopic invariance hypothesis. The only direct test of the transformation properties of the π^0 field (investigation of the reaction $\gamma + \mathrm{d} \to \mathrm{d} + \pi^0$) was discussed in earlier works [9,10], but as was mentioned in [9] this test could not exclude the possibility of descritpion of π^0 by two wave functions φ_3 and φ_0) (*).

I should like to point out a direct experimental test of the existence of the second component π^0 field. The reaction

$$\mathrm{d} + \mathrm{d} \to {}^4\mathrm{He} + \pi^0$$

is trivially forbidden for ordinary mesons by conservation of isotopic spin and allowed for the π_0^0.

The existence of π_0^0 can give the solution to one of the most interesting problems of the physics of elementary particles, the problem of the S-wave

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(9) A. BALDIN and V. MIKCHAÏLOV: *Dokl. Akad. Nauk SSSR*, **91**, 479 (1953).

(10) A. BALDIN: *Suppl. Nuovo Cimento*, **3**, 4 (1956).

(*) Professor R. E. PEIERLS kindly communicated to the author that Y. YAMAGUCHI has also pointed out that at present there exists no experimental data inconsistent with the hypothesis of an isotopic scalar meson. (After writing this paper I have been shown a manuscript of Y. YAMAGUCHI in which he discusses some consequences of different assumptions about the nature of the π_0^0-meson, but there is little overlap between our hypotheses).

pion-nucleon interaction. A discussion of this and other consequences of the existence of the π_0^0 will be given in the next paper.

* * *

It is a pleasure to acknowledge the value of discussions with Professor R. E. PEIERLS. I am also indebted to Mrs. I. EGOROV for help with numerical calculations and to members of the staff of the Department of Mathematical Physics, University of Birmingham for help with the English text of this paper.

RIASSUNTO (*)

È stato fatto l'esame delle sezioni d'urto per le reazioni

$$\gamma + \mathrm{d} \to \begin{cases} 2\mathrm{n} + \pi^+ \\ 2\mathrm{p} + \pi^- \end{cases}$$

in prossimità della soglia. Si presenta il confronto coll'esperienza dei risultati di tale esame e si dimostra che il rapporto negativo/positivo è costante nel campo d'energia fotonica (153÷193) MeV ed uguale a 1.4 (con l'approssimazione del 10%), d'accordo coi risultati della teoria dei campi. Il disaccordo dei precedenti risultati col rapporto di Panofsky e gli spostamenti di fase dello scattering delle onde S si discute dal punto di vista dell'ipotesi del campo dei mesoni π^0. Si propone una prova sperimentale diretta per la verifica di questa ipotesi.

(*) *Traduzione a cura della Redazione.*

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The Interaction between Equilibrium Defects in the Alkali Halides: the «First Excited State» Binding Energies of the Impurity Complex and of the Vacancy Pair.

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and

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(ricevuto il 1° Marzo 1958)

Summary. — The experimentally observed presence in NaCl and KCl crystals of comparable concentrations of complexes between divalent impurity cations and positive-ion vacancies in the ground and first excited states, is shown to be in agreement with the results yielded by the Born-Mayer theory of ionic crystals. Theoretical estimates for the binding energy of the first excited state of the Sr^{2+} impurity complex and of the vacancy pair in NaCl and KCl crystals are presented and discussed.

1. – Introduction.

Divalent impurity cations and cation vacancies, as well as anion and cation vacancies, attract each other in an alkali halide lattice as they bear opposite effective charges, and one may expect a priori that pairs of these oppositely charged point imperfections will be present in the crystal lattice in various states of association, corresponding to various distances of the two interacting defects. While the ground state of these pairs of point defects in the alkali halide crystals has been subjected in recent years to several theoretical in-

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vestigations [1-3], only preliminary results have yet been reported for the excited states [4]. The point is that experimental data have been available for some time on the ground state of these associations, but no experimental method had been found till recently for a comparative study of the various states of association between point defects. Only recently the use of electron spin resonance techniques has allowed the determination in NaCl and KCl crystals of the relative concentrations of the lower association states of complexes between aliovalent paramagnetic impurity cations and cation vacancies [5]. The principle of the method [6] is that a vacancy present in the neighbourhood of a paramagnetic ion reduces the symmetry of the crystalline electric field acting on the ion in a way which depends on the relative position of the two defects in the crystal lattice, and the dependence of the absorption spectrum on the orientation of the crystal relative to the static magnetic field is strictly tied to the symmetry of the crystalline electric field.

In this paper we estimate theoretically the binding energy of the first excited state of the impurity complex between Sr^{2+} ions and positive-ion vacancies in NaCl and KCl crystals: this binding energy turns out to be comparable with the binding energy of the ground state, suggesting the model of a spread out impurity complex. We evaluate also the binding energy of the first excited state of the vacancy pair in the same crystals, which is found to be smaller than the binding energy of the ground state by a few tenths of an eV.

2. – The first excited state of the impurity complex.

While the interaction energy of two charged point defects in the alkali halide lattice, in the limit of large relative distance, is equal to the Coulombic interaction energy of two point charges in a continuous medium characterized by the static dielectric constant of the crystal, the Born-Mayer model of ionic solids yields a value rather less than Coulombic for the binding energy of the ground state of the impurity complex [2] and of the vacancy pair [3] in NaCl

(1) J. R. Reitz and J. L. Gammel: *Journ. Chem. Phys.*, **19**, 894 (1951).

(2) F. Bassani and F. G. Fumi: *Nuovo Cimento*, **11**, 274 (1954).

(3) M. P. Tosi and F. G. Fumi: *Nuovo Cimento*, **7**, 95 (1958).

(4) F. Bassani and F. G. Fumi: *Suppl. Nuovo Cimento*, **1**, 114 (1955).

(5) G. D. Watkins and R. M. Walker: *Bull. Amer. Phys. Soc.*, Ser. II, **1**, 324 (1956), and private communications. The same method has been applied to NaF crystals by W. Hayes (unpublished), and to MgO crystals by J. E. Wertz and P. Auzins: *Phys. Rev.*, **106**, 484 (1957).

(6) E. E. Schneider and J. E. Caffyn: *Report of the Conference on Defects in Crystalline Solids held at Bristol in July* 1954 (London, 1955), p. 74.

and KCl crystals. In fact, the polarization of the material between the two defects increases their interaction energy, while the polarization of the material outside the defects decreases it [7]. On these qualitative grounds one may expect that the binding energy of the first excited state of the impurity complex in these crystals will be nearly equal to the Coulombic value, i.e. comparable with the binding energy of the ground state as calculated by BASSANI and FUMI [2]. Indeed, the polarization of the negative ion sitting between the impurity and the position where the cation vacancy is to be created (Fig. 1) is very large owing to the large polarizability of the ion itself and to the strong electric field due to the impurity ion, and it is very effective in lowering the energy to create the vacancy nearby.

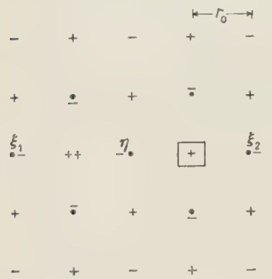

Fig. 1. – The first excited state of the divalent impurity cation-cation vacancy complex in the NaCl lattice. The approximation adopted for the distortion of the lattice around the complex is shown.

The binding energy of the first excited state of the impurity complex can be actually evaluated using the Born-Mayer theory of ionic crystals, as first applied to the study of point imperfections in the alkali halides by MOTT and LITTLETON [8]. This binding energy is given by the difference between the energy to create a positive-ion vacancy in the perfect crystal lattice and the energy to create it in the position of fourth neighbour of a divalent impurity cation. To account for the polarization of the lattice around the vacancy, one evaluates the energy to create a vacancy as the negative of the average of the potential energies in the position of the ion to be removed before and after it has been removed. One needs therefore the distortion of the lattice around the divalent impurity cation and around the impurity complex in the first excited state (Fig. 1). The displacement ξr_0 of the ions which are nearest neighbours of the isolated impurity cation is determined by imposing the equilibrium condition to the electrostatic and repulsive forces acting on each ion [8.1]. The displacements $\xi_1 r_0$ and $\xi_2 r_0$ of the ions which are nearest neighbours respectively of the impurity cation and of the cation vacancy tied in the complex are determined in the same way by considering only the effect of the nearest defect but neglecting the long range polarization since the complex is neutral [8.1]. Finally, the displacement ηr_0 of the negative ion sitting between the impurity and the vacancy is determined by imposing the equilibrium condition to the electrostatic and repulsive forces acting on

(7) F. G. FUMI and M. P. TOSI: *Faraday Soc. Disc.*, **23**, 92 (1957).
(8) N. F. MOTT and M. J. LITTLETON: *Trans. Faraday Soc.*, **34**, 485 (1938).

it, due to the two defects and to the polarization induced by the defects on the nearest ions. When the lattice distortion around the impurity and around the complex in the first excited state is known, the energy to create a cation vacancy in the position of fourth neighbour of the impurity can be easily evaluated following the procedure already applied in the analogous problem of the ground state for the impurity complex [1,2]. The numerical results obtained for Sr^{2+} ions in NaCl and KCl crystals are collected in Table I. For the constants entering the calculation we have used the numerical values quoted in ref. [2], Table I, except that we adopt for the electronic polarizabilities the values reported by TESSMAN, KAHN and SHOCKLEY [9] for ions in crystal lattices [10]. The repulsive form given by BORN and MAYER [11,2] has been used.

TABLE I. – *Binding energy of the* Sr^{2+} *impurity complex in the first excited state in* NaCl *and* KCl *crystals and related ionic displacements.*

	ξ	ξ_1	ξ_2	η	Binding energy (eV)
NaCl	0.033	0.042	0.086	0.090	$4.78 - 4.37 = 0.41$
KCl	0.099	0.109	0.084	0.164	$4.44 - 3.95 = 0.49$

The approximation adopted for the distortion of the lattice around the complex is probably the principal source of error in our calculation. However, it has been verified that reasonable changes of the displacements around the quoted values have a small effect on the binding energy: this is true also for the displacement ηr_0, which may be affected by a fairly large error owing also to the uncertainties of the repulsive interaction law at short distances [7]. The dipole-dipole interaction has been found to give a substantial (negative) contribution to the binding energy of the first excited state of the complex, contrary to the case of the ground state where it can be neglected [1]: this contribution is substantially greater in NaCl than in KCl.

The binding energies in NaCl and KCl crystals between a Sr^{2+} ion and a cation vacancy in the first excited state (0.41 eV in NaCl; 0.49 eV in KCl)

(9) J. R. TESSMAN, A. H. KAHN and W. SHOCKLEY: *Phys. Rev.*, **92**, 890 (1953).

(10) This entails some numerical changes in the displacements (cfr. Table I and ref. [2], Tables III and IV) and in the energies to create cation vacancies (for the energy to create cation vacancies in the perfect crystal lattice, see ref. [3], Sect. **3**). However, the values reported by BASSANI and FUMI [2] for the binding energy of the ground state of the impurity complex remain practically unchanged (see ref. [3], footnote [16]).

(11) M. BORN and J. MAYER: *Zeits. f. Phys.*, **75**, 1 (1932); J. MAYER and H. HELMHOLZ: *Zeits. f. Phys.*, **75**, 19 (1932).

are found to be comparable with the binding energies in the ground state (0.43 eV in NaCl; 0.38 eV in KCl) (ref. [3], footnote [16]), the ground state being slightly more strongly bound in NaCl and less strongly bound in KCl. These results are in good qualitative agreement with the experiments by WATKINS and WALKER [5], who observe by paramagnetic resonance techniques an inversion in the relative binding energy of the ground and first excited state for the Mn^{2+} impurity complex in going from NaCl to KCl crystals: the ground state is more strongly bound in NaCl, and the difference in binding energy between the two states is quite small in both salts, of the order of a few hundredths of an eV [12]. Moreover, our model of a spread out impurity complex can explain qualitatively ASCHNER's observations which indicate that the associated cation vacancies are as effective as the free vacancies in aiding the self-diffusion of Na^+ ions in Cd^{2+}-doped NaCl [13,14].

It is also worth noting that the value yielded by our lattice calculation for the binding energy of the first excited state of the Sr^{2+} impurity complex is comparable with the Coulombic value (0.46 eV in NaCl; 0.49 eV in KCl), so that the approximation adopted by LIDIARD [15] in the analysis of the ionic conductivity of the alkali halides doped with divalent impurity cations appears justified.

3. – The first excited state of the vacancy pair.

To calculate the binding energy of the first excited state of the vacancy pair, one has to evaluate the energy to extract a positive ion from the position of third neighbour of a negative-ion vacancy, and to subtract it from the energy to extract a positive-ion from the perfect crystal lattice. The determination of the distortion of the lattice around the vacancy pair in its first excited state is straightforward: the outward displacements $\xi_2 r_0$ and $\xi_3 r_0$ of the nearest neighbours of the positive- and negative-ion vacancy are evaluated as the displacements of the nearest neighbours of the isolated vacancies, neglecting the long range polarization. The numerical results are collected in Table II. It can be noted that the Born-Mayer model yields for the binding energy of the first excited state of the vacancy pair a value

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([12]) It may also be expected that in NaF, where the polarizability of the anion is considerably smaller than in NaCl, the ground state of association should be more strongly bound, as observed by HAYES (private communication).

([13]) J. F. ASCHNER: *Self-Diffusion in Sodium and Potassium Chloride*, Ph. D. Thesis (University of Illinois, 1954).

([14]) A. B. LIDIARD: *Handb. d. Phys.*, **20**, 246 (1957), and private communication.

([15]) A. B. LIDIARD: *Phys. Rev.*, **94**, 29 (1954).

TABLE II. – *Binding energy of the first excited state of the vacancy pair in* NaCl *and* KCl *crystals and related ionic displacements.*

	$\xi^{\boxminus}$ (*)	ξ_2	ξ_3	Binding energy (eV)
NaCl	0.104	0.086	0.129	4.78 — 4.50 = 0.28
KCl	0.076	0.084	0.095	4.44 — 4.06 = 0.38

(*) The values of the quantity $\xi^{\boxminus} r_0$, which is the displacement of the nearest neighbours of an isolated anion vacancy, are given by TOSI and FUMI (ref. 3, Table I).

substantially smaller than for the binding energy of the ground state (0.60 eV in NaCl; 0.72 eV in KCl) (3). Moreover, the value is still definitely smaller than the Coulombic value (0.53 eV in NaCl; 0.57 eV in KCl), as can be easily understood on the basis of the qualitative argument given at the beginning of Sect. 2.

* * *

The authors are indebted to Professor F. G. FUMI, who suggested the problem, for helpful discussions and continued encouragement. Very informative correspondence with Dr. G. D. WATKINS is acknowledged.

RIASSUNTO

Si mostra che la presenza in cristalli di NaCl e KCl di concentrazioni confrontabili di complessi fra cationi d'impurità bivalenti e posti vacanti da ione positivo negli stati fondamentale e primo eccitato, osservata sperimentalmente, è in accordo con i risultati forniti dalla teoria di Born-Mayer dei cristalli ionici. Sono presentati e discussi risultati teorici per l'energia di legame del primo stato eccitato del complesso di impurità per ioni Sr^{2+} e della coppia di posti vacanti in cristalli di NaCl e KCl.

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Charge Conservation in the Bethe-Salpeter Equation.

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(ricevuto il 5 Marzo 1958)

Summary. — The scalar product between two bound states is expressed in terms of a conserved charge, and reduced to a standard covariant form.

1. – Introduction.

In a previous paper (1), hereafter referred to as I, one of us has derived a general formula expressing the scalar product between two bound states in terms of their Bethe-Salpeter wave functions. Mandelstam (2) has derived an alternative form for the scalar product, applicable to states where there is a non-zero conserved charge (e.g. electric charge, nucleon charge, etc.), and this form has been used by various authors (3).

The two forms for the scalar product are quite different in appearance, though of course formal considerations indicate that they should be equivalent. In the present paper we treat a system in which the conserved charge is carried by one or more particles, and trace out explicitly the equivalence between the scalar product derived from charge quantization and that given in I. We also indicate that when the Bethe-Salpeter kernels and one-particle propagators are approximated by dropping out various graphs the equivalence

(1) G. R. Allcock: *Phys. Rev.*, **104**, 1799 (1956).

(2) S. Mandelstam: *Proc. Roy. Soc.* (*London*), **A 233**, 248 (1955).

(3) K. Nishijima: *Progr. Theor. Phys.*, **13**, 305 (1955); A. Klein and C. Zemach: *Phys. Rev.*, **108**, 126 (1957); F. L. Scarf and H. Umezawa: *Phys. Rev.*, **109**, 1848 (1958).

persists, provided a proper correspondence between graphs is maintained between the two methods.

2. – Matrix elements of the charge.

We consider two bound states $\langle b|$ and $|a\rangle$, described by Bethe-Salpeter wave functions $\overline{\chi}_b$ and χ_a, as in I. We suppose, for the sake of the argument, that the conserved current j^μ is carried by a Dirac field ψ, so that it takes the form $\overline{\psi}\gamma^\mu\psi$. The current matrix element is then

$$\langle b|j^\mu(x)|a\rangle = \langle b|\overline{\psi}(x)\gamma^\mu\psi(x)|a\rangle\,. \tag{1}$$

MANDELSTAM (2) has shown that it can be written covariantly in the form

$$\langle b|j^\mu(x)|a\rangle = \overline{\chi}_b R^\mu(x)\chi_a\,. \tag{2}$$

Here we imply integrations over the space-time arguments of the Bethe-Salpeter wave functions $\overline{\chi}_b$ and χ_a. These space-time arguments appear also in the graphic expression $R^\mu(x)$, which arises as follows.

To express the current matrix element (1) in the form (2) one augments the operator arguments of the propagator G descriptive of the bound system, by putting in the current operator $\overline{\psi}(x)\gamma^\mu\psi(x)$. Those graphs belonging to this new propagator and contributing to (1) may be enumerated one-to-one by inserting a vertex γ^μ in turn into each ψ-line of the graphs for G. Since G has the structure expressed by the equation

$$G = G_0 + G_0 K G\,, \tag{3}$$

the propagator with current has the structure

$$G R^\mu(x) G\,, \tag{4}$$

where $R^\mu(x)$ in turn can be written as a sum of two terms,

$$R^\mu(x) = K^\mu(x) + K G_0^\mu(x) K\,. \tag{5}$$

Here $K^\mu(x)$ and $G_0^\mu(x)$ are the expressions arising from vertex insertions in the ψ lines of K and G_0 respectively.

Inserting (5) into (2) and integrating over all space we obtain the matrix element of the total charge,

$$\langle b|Q(t)|a\rangle = \int \mathrm{d}^3\boldsymbol{x}\,\overline{\chi}_b\big(K^0(x) + K G_0^0(x) K\big)\chi_a\,. \tag{6}$$

Mandelstam's [2] scalar product is obtained by exploiting the usual relationship between the charge operator $Q(t)$ and its integer eigenvalue n. Thus for the scalar product one should have

$$\langle b | a \rangle = \frac{1}{n} \langle b | Q(t) | a \rangle \,, \tag{7}$$

where $\langle b | Q(t) | a \rangle$ is given by (6).

3. – Simplification of the charge matrix elements.

If we analyse $\bar{\chi}_b$ and χ_a into momentum eigenstates, the charge matrix element (6) appears as a sum of contributions, one from each momentum eigenvalue. For the space integration in (6) gives momentum conservation. It follows that when the states $|a\rangle$ and $|b\rangle$ have the same mass eigenvalue M the time dependence from χ_a in (6) is completely compensated by that in $\bar{\chi}_b$, so that (6) becomes explicitly independent of the time t, irrespective of the formal theorem of charge conservation. Naturally our interest in the present work is confined to this case, since with unequal masses $\langle b | a \rangle$ vanishes by space-time translational invariance.

In view of this explicit time independence it is legitimate to replace (6) by its average over a long time interval $(T, -T)$; thus

$$\langle b | Q | a \rangle = \frac{1}{2T} \int_{-T}^{T} \mathrm{d}^4 x \bar{\chi}_b \{ K^0(x) + K G_0^0(x) K \} \chi_a \,. \tag{8}$$

The above expression corresponds in some sense to a charge measurement in which the energy of the system is completely undisturbed, and leads to considerable simplifications. For the four-dimensional integrations on x in (8) allow us to apply Ward's equation

$$S_F(p) \gamma^\mu S_F(p) = - \partial S_F(p) / \partial p_\mu \,, \tag{9}$$

or

$$\int S_F(x_1 - x) \, \mathrm{d}^4 x \gamma^\mu S_F(x - y_1) = - (x_1^\mu - y_1^\mu) i S_F(x_1 - y_1) \,. \tag{10}$$

In the matrix element (8) we do not actually encounter the expression (10), but rather the expression

$$\frac{1}{2T} \int_{-T}^{T} i S_F(x_1 - x) \, \mathrm{d}^4 x \gamma^\mu i S_F(x - y_1) \,. \tag{11}$$

Here x_1 and y_1 are the end points of a section of a ψ-line into which a vertex γ^μ has been inserted. To see how to deal with (11), we must consider the integrations over the space-time co-ordinates other than x in (8). These integrations extend over the whole of space-time, but for any pair of states $|b\rangle$ and $|a\rangle$ we will be able to find some large but finite time interval 2τ surrounding x^0, outside which the integrand is effectively zero. This being so, the points x_1 and y_1 with which we have to deal in (11) will be separated in time by an amount of order 2τ, or less. If then we take $T \gg \tau$, the time integration in (11) usually extends far to each side of the interval (x_1^0, y_1^0), and the contribution of the pair of points x_1, y_1 can then be calculated by using (10). Therefore we may with sufficient accuracy replace the expression (11) by

$$(12)\qquad \begin{cases} \dfrac{1}{2T}(x_1^\mu - y_1^\mu) iS_F(x_1 - y_1)\,, & x_1 \text{ and } y_1 \text{ both within } (T, -T)\,, \\ 0, & x_1 \text{ and } y_1 \text{ not both within } (T, -T)\,. \end{cases}$$

The substitutions (12) are correct apart from edge effects arising when x_1 or y_1 lie within a time range τ of the limits $\pm T$. The edge contributions appear multiplied by a factor $1/2T$ from (11), and therefore vanish as τ/T as $T \to \infty$. Hence, if T/τ is sufficiently large, the integrals on x in (8) may be performed by replacing in turn each ψ-line factor $iS_F(x_1 - y_1)$ in K or G_0 by (12). Thus, neglecting edge effects, each vertex insertion brings in a factor $(1/2T)\cdot(x_1^\mu - y_1^\mu)$ if x_1^0 and y_1^0 lie within $(T, -T)$, and a factor zero elsewhere.

Again bearing in mind the localization within the range τ, we see that to within edge contributions the factors $x_1^\mu - y_1^\mu$ sum to zero for vertex insertion in closed loops, while for any line running right through a graph they sum to the difference of the end co-ordinates of the line.

Evidently for the charge matrix element (8) we now have the expression

$$(13)\quad \langle b|Q|a\rangle = \sum \lim_{T\to\infty} \frac{1}{2T} \int_T^T \mathrm{d}^4x_1 \int_T^T \mathrm{d}^4y_1 \{\bar\chi_b(\dots x_1 \dots)(x_1^0 - y_1^0) K(\dots x_1 \dots; \dots y_1 \dots)\cdot$$

$$\cdot \chi_a(\dots y_1 \dots) + \bar\chi_b(\dots) K(\dots; \dots x_1 \dots)(x_1^0 - y_1^0) G_0(\dots x_1 \dots; \dots y_1 \dots) K(\dots y_1 \dots; \dots)\chi_a(\dots)\}\,.$$

Here we understand that K and G_0 are decomposed graphically so that the charge-carrying lines can be seen. The co-ordinates x_1 and y_1 denote the end points of a typical charge-carrying line, and $\sum$ denotes a summation over all such lines. Integrations over the whole of space-time are understood for the co-ordinates other than x_1 and y_1.

At this stage it is useful to introduce centre co-ordinates X^μ, Y^μ and re-

lative co-ordinates r_i^μ, s_i^μ thus:

$$(14)\qquad \begin{cases} X^\mu = \sum\limits_i \alpha_i x_i{}^\mu\,, \qquad Y^\mu = \sum\limits_i \alpha_i y_i{}^\mu\,, \qquad \sum\limits_i \alpha_i = 1\,, \\ r_i{}^\mu = x_i{}^\mu - X^\mu\,, \qquad s_i{}^\mu = y_i{}^\mu - Y^\mu\,. \end{cases}$$

Here the α_i are arbitrary real numbers introduced only as an aid in the calculations. We now substitute

$$(15)\qquad x_1^0 - y_1^0 = X^0 + r_1^0 - (Y^0 + s_1^0)\,,$$

and apply the Bethe-Salpeter equations $\chi = G_0 K \chi$, $\bar\chi = \bar\chi K G_0$ to the second term on the R.H.S. of (13). Thus we have, for example,

$$(16)\qquad r_1^0 \int\limits_{-T}^{T} \mathrm{d}^4 y_1 G_0(\dots x_1 \dots; \dots y_1 \dots) K(\dots y_1 \dots; \dots) \chi_a(\dots) = \\ = r_1^0 \{ \chi_a(\dots x_1 \dots) + \text{edge terms} \}\,.$$

The edge terms above arise only when x_1^0 lies within a range τ of $\pm T$, and they are multiplied by the factor r_1^0, which is confined to within the range $(\tau, -\tau)$ by the wave function χ_a. Evidently these edge terms can be neglected, because of the factor $1/2T$ in (13). A similar application of the Bethe-Salpeter equation to the term $X^0 \int G_0 K \chi_a$ is useless, giving non-vanishing edge errors because X^0 is of order $\pm T$ in the edge regions. Substituting (15) in the first term on the R.H.S. of (13), and (16) in the second term, the contributions of the relative co-ordinates r and s cancel between the first and second terms in the limit $T \to \infty$, and we are left with

$$(17)\qquad \langle b | Q | a \rangle = \sum \lim_{T\to\infty} \frac{1}{2T} \int\limits_{-T}^{T} \mathrm{d}^4 x_1 \int\limits_{-T}^{T} \mathrm{d}^4 y_1 \{ \bar\chi_b(\dots x_1 \dots)(X^0 - Y^0) \cdot \\ \cdot K(\dots x_1 \dots; \dots y_1 \dots) \chi_a(\dots y_1 \dots) + \bar\chi_b(\dots) K(\dots; \dots x_1 \dots)(X^0 - Y^0) \cdot \\ \cdot G_0(\dots x_1 \dots; \dots y_1 \dots) K(\dots y_1 \dots; \dots) \chi_a(\dots) \}\,.$$

The contributions to (17) come from the region $-\tau < X^0 - Y^0 < \tau$. This being so, we may with negligible edge error replace

$$\int\limits_{-T}^{T} \mathrm{d}^4 x_1 \int\limits_{-\infty}^{\infty} \mathrm{d}^4 x_2 \int\limits_{-\infty}^{\infty} \mathrm{d}^4 x_3 \dots$$

in (17) by

$$\int\limits_{-T}^{T} \mathrm{d}^4 X \int\limits_{-\infty}^{\infty} \mathrm{d} V\,,$$

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where

$$\mathrm{d}V \equiv \delta^4\big(\sum_i r_i^\mu\big)\, \mathrm{d}^4 r_1\, \mathrm{d}^4 r_2\, \mathrm{d}^4 r_3 \,\ldots\, . \tag{18}$$

The y integrations can be treated similarly. When this is done the summation over the charge carrying lines becomes trivial, since the factors $X^0 - Y^0$ are the same for all lines. If there are $n+m$ forward directed charge-carrying lines and m backward directed charge-carrying lines the summation brings simply a factor n. Since the formal theorem of charge quantization ascribes to $\langle b|Q|a\rangle$ the value $n\langle b|a\rangle$, we obtain now for the scalar product ($n \neq 0$),

$$\begin{aligned}\langle b|a\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathrm{d}^4X\int_{-T}^{T}\mathrm{d}^4Y\{\bar\chi_b(\ldots x \ldots)(X^0 - Y^0)K(\ldots x \ldots; \ldots y \ldots)\cdot\\ \cdot\chi_a(\ldots y \ldots) + \bar\chi_b(\ldots)K(\ldots; \ldots x \ldots)(X^0 - Y^0)\cdot\\ \cdot G_0(\ldots x \ldots; \ldots y \ldots)K(\ldots y \ldots; \ldots)\chi_a(\ldots)\}\, .\end{aligned} \tag{19}$$

Integrations over the elements $\mathrm{d}V$ (eq. (18)) accompany the X and Y integrations.

In a theory in which all propagators and wave functions are defined with respect to renormalized Heisenberg operators $Z_\psi^{\frac{1}{2}}\psi$, eq. (19) still holds, for $\langle b|Q|a\rangle$ then should have the value $Z_\psi^{-1}n\langle b|a\rangle$, while vertex insertion brings a factor $(1/2T)Z_\psi^{-1}(x_1^\mu - y_1^\mu)$ because of the renormalization vertices $- i(1 - Z_\psi)\cdot\cdot(i\gamma\partial - m)$.

The integrations on the centre co-ordinates in the scalar product (19) must now be performed. Since $X^0 - Y^0$ is confined to a range $2\tau \ll 2T$ by the wave functions χ_a and $\bar\chi_b$, we may replace $\int_{-T}^{T}\mathrm{d}X^0\int_{-T}^{T}\mathrm{d}Y^0$ by $\int_{-T}^{T}\mathrm{d}X^0\int_{-\infty}^{\infty}\mathrm{d}(X^0 - Y^0)$ with vanishing edge error. If we now perform all the integrations other than that on X^0, the translational invariance ensures momentum conservation, so that the frequencies in $\bar\chi_b$ annul those in χ_a, giving an integrand which is independent of the integration variable X^0 except in the edge region. The integration on X^0 gives therefore merely a factor $2T$ to cancel the factor $1/2T$ in (19).

To express this feature explicitly we introduce Fourier transforms for G_0, K, $\bar\chi_b$ and χ_a, as in I, thus

$$\left\{\begin{aligned} &G_0(\ldots x \ldots; \ldots y \ldots) = (2\pi)^{-4}\int \mathrm{d}^4p\, G_0(p; \ldots r \ldots; \ldots s \ldots)\exp[-ip(X - Y)]\, ,\\ &\chi_a(\ldots x \ldots) = (2\pi)^{-3}\int(\mathrm{d}^3P/2P^0)\chi_a(P; \ldots r \ldots)\exp[-iPX]\, .\end{aligned}\right. \tag{20}$$

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Using the above Fourier transforms, and suppressing the relative co-ordinates, the scalar product (19) now becomes

$$(21)\qquad \langle b|a\rangle = \frac{-i}{(2\pi)^3}\int \frac{d^3P}{2P^0}\,\bar{\chi}_b(P)\,\frac{1}{2P^0}\left\{\frac{\partial K(P)}{\partial P_0} + K(P)\,\frac{\partial G_0(P)}{\partial P_0}\,K(P)\right\}\chi_a(P)\,.$$

The factor $\partial K(P)/\partial P_0$ arises because we deal with the Fourier transform of $(X^0-Y^0)K(X-Y;\,...r...;\,...s...)$ which is $-i\partial K(p;\,...\ r\ ...;\,...\ s\ ...)/\partial p_0$. The factor $\partial G_0(P)/\partial P_0$ arises in the same way.

An expression very similar to (21) was obtained by a different method in I. In the following section we shall show that the two expressions are, in fact, explicitly equivalent. We may close the present section by noting that our proof of the equality of the scalar product (7) and the expression (21) is valid even when one approximates G_0 and K by using subsets of the full series of graphic contributions. It entails only that one take just those graphs for the current density j^μ which arise from vertex insertion in the graphs actually used to represent G_0 and K. For this reason computations made on the basis of charge quantization will necessarily agree exactly with any based on (21), provided the Bethe-Salpeter equations $\chi = G_0K\chi$ are solved exactly with the approximated G_0 and K (*).

4. – Lorentz invariance of the scalar product.

The method used in I to determine the scalar product rested essentially on finding an expansion of the propagator $G(p;\,...r...;\,...s...)$ in the neighbourhood of the singularity at $p^2=M^2$, M denoting the mass of the bound system. The value of $G(p)$ at a point p just off the mass hyperboloid was defined by eq. (21) of that paper, which we shall quote as (I.21). The Bethe-Salpeter equation for G appeared in the form

$$(\mathrm{I}.22)\qquad g(p)\equiv(p^2-M^2)G(p) = (p^2-M^2)G_0(p) + G_0(p)K(p)g(p)\,.$$

In I the variation of the regular function $g(p)$ across the mass hyperboloid was investigated by writing $p^\mu = P^\mu(1+\varepsilon)$, where $P^2=M^2$. We proceed here more generally, by considering an arbitrary displacement δp^μ, so that on differentiating (I.22) above we replace (I.25) by the four equations

$$(22)\qquad \{1-G_0(P)K(P)\}\partial g(P)/\partial P^\mu = 2P_\mu G_0(P) + (\partial\{G_0(P)K(P)\}/\partial P^\mu)g(P)\,.$$

(*) KLEIN and ZEMACH [3] have examined the form of the current density (2) in the lowest ladder approximation. We wish to point out that their scalar product (95), obtained for this case, contains one too many integrations on the centre co-ordinates. The dependence on the relative co-ordinates expressed by eq. (96) is correct, however.

As in I, we express $g(P; ...r...; ...s...)$ as the linear combination

$$g(P; ...r...; ...s...) = \sum_{\alpha\beta} u_\alpha(P; ...r...)\lambda_{\alpha\beta}(P)\bar{v}_\beta(P; ...s...) , \tag{23}$$

where u_α and $\bar{v}_\beta$ are complete sets of linearly independent solutions of the Bethe-Salpeter equations

$$u(P) = G_0(P)K(P)u(P) , \qquad \bar{v}(P) = \bar{v}(P)K(P)G_0(P) .$$

Then, as noted in I, eqs. (22) serve to determine the matrix of coefficients $\lambda_{\alpha\beta}$. For the operator $1 - G_0K$ has zero determinant, so that (22) can only be solved for $\partial g(P)/\partial P^\mu$ if pre-multiplication of its R.H.S. by the null vectors $\bar{v}_\beta(P)K(P)$ gives zero. Hence, using (23), we obtain four sets of equations, replacing (I.27). They are

$$N_{\mu\alpha\beta}(P)\lambda_{\beta\gamma}(P) + 2P_\mu\delta_{\alpha\gamma} = 0 , \tag{24}$$

where

$$N_{\mu\alpha\beta}(P) \equiv \bar{v}_\alpha(P)\big(K(P)\{\partial G_0(P)/\partial P^\mu\}K(P) + \partial K(P)/\partial P^\mu\big)u_\beta(P) . \tag{25}$$

Compatibility of these four sets of equations follows from the fact that $N_{\mu\alpha\beta}(P)$ is *invariably* parallel to P_μ, as we shall presently show,

$$N_{\mu\alpha\beta}(P) \equiv P_\mu c_{\alpha\beta}(P) , \tag{26}$$

so that (24) reduce to a single scalar equation

$$c_{\alpha\beta}(P)\lambda_{\beta\gamma}(P) + 2\delta_{\alpha\gamma} = 0 . \tag{27}$$

The matrix equation (27) provides a convenient specification of the coefficients $\lambda_{\alpha\beta}$ at the pole of the propagator, though it remains to be seen whether λ actually fulfills the requirement that the quantization metric is positive definite.

To prove (26) we invoke the Lorentz covariance of the theory. This asserts that within any given frame of reference we may attribute to each infinitesimal Lorentz transformation of P^μ, $P^\mu \to P^\mu + \delta P^\mu$, a corresponding transformation $u_\beta(P; ...s...) \to u_\beta(P; ...s...) + \delta u_\beta(P; ...s...)$. Thus, writing down the Bethe-Salpeter equation for $u_\beta + \delta u_\beta$ we have,

$$u_\beta + \delta u_\beta = (G_0K + \{\partial(G_0K)/\partial P^\mu\}\delta P^\mu)(u_\beta + \delta u_\beta) , \tag{28}$$

whence

$$(1 - G_0K)\,\delta u_\beta = \{\partial(G_0K)/\partial P^\mu\}\delta P^\mu \cdot u_\beta . \tag{29}$$

Premultiplying (29) by $\bar{v}K$, and using the Bethe-Salpeter equation for $\bar{v}$, and noting that δP^μ is any infinitesimal four-vector orthogonal to P^μ, we at once obtain (26). It will be seen that this result is also independent of the sub-sets of graphs taken to represent G_0 and K.

Returning to the normalization integral, we require the matrix λ^{-1}, which from (27) is given by

$$\lambda^{-1}{}_{\alpha\beta} = -\tfrac{1}{2}c_{\alpha\beta}\,. \tag{30}$$

We can now evaluate (30) in various ways, the equivalence of which follows from the identity (26). For instance, if we take the zero'th component of (26) we obtain

$$\lambda^{-1} = -N^0/2P^0\,, \tag{31}$$

while if we contract (26) with the four-vector P_μ (assuming $M^2 \equiv P^2 > 0$) we obtain the more obviously covariant expression

$$\lambda^{-1} = -N^\mu P_\mu/2M^2\,. \tag{32}$$

At this point we must refer the reader to Sect. **5** of I, where the scalar product is expressed in terms of the matrix λ^{-1} by eq. (I.35). Substitution of (31) into (I.35) verifies at once our expression (21) for the scalar product, while substitution of (32) gives the obviously covariant expression (I.37) below:

$$\langle b|a\rangle = \frac{-i}{(2\pi)^3}\int\frac{\mathrm{d}^3P}{2P^0}\,\bar{\chi}_b(P)\,\frac{P_\mu}{2M^2}\left\{\frac{\partial K(P)}{\partial P_\mu} + K(P)\,\frac{\partial G_0(P)}{\partial P_\mu}\,K(P)\right\}\chi_a(P)\,. \tag{I.37}$$

The scalar product (21) derived from charge quantization is obviously entirely equivalent to the general covariant form (I.37), by virtue of the identity (26).

RIASSUNTO (*)

Il prodotto scalare tra due stati legati è espresso in termini di una carica conservata e ridotto in forma covariante convenzionale.

(*) *Traduzione a cura della Redazione.*

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On the Σ^- Capture and the Λ-Nucleon Cross-Section.

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Summary. — An interpretation of the data on the capture of Σ^--particles by nuclei of photoemulsion has been attempted on the basis of the independent particle model. In particular it has been found that the probability of the Λ-particle produced in the capture to remain trapped in the nucleus, and hence the average size of the Σ^- capture star, strongly depends on the mean free path of Λ-particles in nuclear matter. In order to justify the experimental data the Λ-N elastic cross-section at energies around 50 MeV should be about 15 mb.

1. – Introduction.

Various calculations based on the independent particle model and with the Goldberger method, performed either analytically or with the help of the Monte-Carlo method, have successfully accounted for the mechanism of nuclear disintegrations induced by fast protons (1) and by the capture of π^- and K^- mesons (2,3). Similar calculations have also been found useful for deriving information on the interaction properties of «strange particles» with nucleons (4), from the analysis of the more easily available experimental data on complex nuclei.

(1) G. Bernardini, E. T. Booth and S. J. Lindenbaum: *Phys. Rev.*, **83**, 669 (1951).

(2) V. De Sabbata, E. Manaresi and G. Puppi: *Nuovo Cimento*, **10**, 1704 (1953).

(3) F. C. Gilbert, C. E. Violet and R. S. White: *Phys. Rev.*, **107**, 228 (1957). R. H. Capps: *Phys. Rev.*, **107**, 239 (1957).

(4) See for instance the papers of the Göttingen group on K^+ and K^- nuclear interactions as well as of the Bologna, Bristol-Dublin and Padua groups on K^+ interaction: *Nuovo Cimento* and *Zeits. f. Naturfor.* (1956-1957).

This encouraged us to apply the same method for the nuclear capture of Σ^- hyperons. The purpose was to give an account of the experimental data on this type of events and, more specifically, to derive some information about the nuclear interaction of Λ particles of which the Σ^- capture process constitutes a convenient source.

2. – Disintegration induced by the Σ^- capture.

The capture of a Σ^- particle by a nucleus will occur from a state of orbital angular momentum which depends on the cross-section of the Σ-nucleon interaction. For the present calculation we have assumed that the cross-section for the process $\Sigma+N \to \Lambda+N$ is rather small ($\sim \frac{1}{5}$ geometric, as indicated by the work of FERRARI and TONIN [5] and of ALLES [6] so that the Σ^- is able to penetrate inside the nucleus before being actually absorbed. Consequently we have taken the points of capture to be uniformly distributed inside the nucleus. It can be seen however that a change of this distribution, for instance corresponding to a Coulomb potential oscillator or to capture occuring only at the periphery plays little role in the final results of the calculations.

The Σ-particle may be captured following the two reactions:

a) $$\Sigma^- + p \to \Lambda + n + 81\ \text{MeV}$$

b) $$\Sigma^- + p \to \Sigma^0 + n + 8\ \text{MeV}$$

plus, eventually, reactions involving two nucleons. Since in the latter case both the nucleons have to be emitted with energies higher than the maximum-Fermi energy, these reactions are likely to be strongly depressed by the Pauli exclusion principle. For this reason, and also in analogy with K-meson capture [7], two nucleon reactions have been considered rare and therefore neglected. Of the two reactions involving a single nucleon, reaction *b*) is found to be forbidden in about 90% of the cases by the Pauli principle because of the low Q-value. Further, out of the two subsequent reactions:

c) $$\Sigma^0 \to \Lambda + \gamma$$

and

d) $$\Sigma^0 + N \to \Lambda + N$$

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[5] F. FERRARI and M. TONIN: *Proc. of Padua-Venice Conference* (1957).

[6] W. ALLES: private communication.

[7] According to different groups the frequency of two-nucleon K^- captures lies between 3 and 10 percent. (See *Padua-Venice Conference report*).

of the Σ^0 created inside the nucleus, the second one is likely to be dominant. Since the energy spectrum of the Λ-particles emitted from the reaction *d*) will be practically the same as in the reaction *a*), we may consider only the latter one as responsible for the development of the Λ cascade inside the nucleus.

The spectrum of Λ-particles emitted in reaction *a*) has been computed with essentially the same method used by CAPPS (8) for obtaining the spectrum of Σ-particles in the reaction

$$K^- + p \to \Sigma^\pm + \pi^\mp .$$

The depth of potential well, V, of the Λ-particle has been taken to be 17 MeV, as resulting from the formula

$$V = \varrho(3U_p + U_a)/4 ,$$

where ϱ is the nuclear density and $U_{p,a}$ are the volume integrals of the Λ-N interaction potential for parallel and anti-parallel spins. For these we have taken the values $84 \cdot 10^{-39}$ and $403 \cdot 10^{-39}$ MeV cm³ calculated by DALITZ (9) from the binding energies of light hyperfragments. The energy spectrum of the Λ-particles inside the nucleus is given by the curve of Fig. 1.

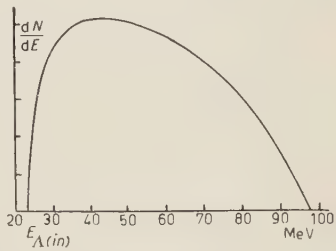

Fig. 1. - Energy spectrum at production of Λ-particles inside the nucleus.

From the point of capture we have then to trace both the nucleon and the Λ-particle up to the nuclear surface, where they will either escape or be trapped if the kinetic energy is lower than the corresponding potential step; this has been taken as energy independent.

Due to its great complexity this part of the calculation has been done with the help of the Monte-Carlo method. For the cascade induced by the neutron, the values of the various parameters (σ_{tot}, σ_{diff}, etc.) have been taken to be the same as in reference (2); for the Λ cascade the Λ-N scattering in the center of mass system has been assumed to be isotropic and energy independent, the total cross-section on free nucleons, $\sigma_{\Lambda\text{-N}}$, being left as a free parameter.

Four sets of calculation have been performed with different values of $\sigma_{\Lambda\text{-N}}$, namely 0, 15, 30 and 60 mb. The cross-sections on protons and neutrons have been taken to be equal as expected from the fact that the isotopic spin of

(8) R. H. CAPPS: see ref. (3).
(9) R. H. DALITZ: *Rep. on Progr. in Phys.*, **20**, 252 (1957).

the Λ is zero. The calculation has been done for a nucleus with $A = 100$.

After the nuclear cascade, the nucleus will be generally left in an excited state and nuclear particles will evaporate. In addition also the Λ-particle previously trapped may evaporate; we have however reasons to believe that this will occur rather infrequently. Indeed, if the Λ-particle has a non-vanishing cross-section with nucleons, for instance around 10 mb, after being trapped it will collide with a number of nucleons in a time of the order of 10^{-21} s. The effect of each collision will be, as a consequence of the Pauli exclusion principle acting only on the nucleon, to reduce the Λ energy; therefore in a very short time, the Λ-particle will almost reach its ground state. Its subsequent re-acceleration and evaporation should then be even less probable than the evaporation of a given neutron of the nucleus: a process which at the excitation energies here concerned seems to take place with a probability of only about 5%. This reasoning has led us to neglect the evaporation of the Λ-particle and only to treat that of nucleons; this has been done with the same method as in reference (2).

After a time much longer than the evaporation process the trapped Λ-particle will undergo decay. In a heavy nucleus mesonic decay should be practically absent and the dominant reactions will be

$$e) \qquad \Lambda + n \rightarrow n + n$$

and

$$f) \qquad \Lambda + p \rightarrow n + p\,.$$

Experimental data on the relative frequencies of reactions *e*) and *f*) are still scarce for light nuclei (10) and totally absent for the heavy ones. In our calculations we have assumed the two reactions to occur with equal probability; for $\sigma_{\Lambda\text{-}N} = 15$ mb additional computations have been made for the cases of decay « stimulated » exclusively by protons, or by neutrons.

The two nucleons emitted from the reactions *e*) and *f*) have been followed through the nucleus and the emission of particles due to this second cascade and in the consequent evaporation has been computed.

The calculations have been done mostly graphically with the help of a device which permits the construction of the model of a three dimensional nuclear cascade. For the cascades initiated by a nucleon as well as for computing the emission of evaporated particles, we have had the opportunity of utilizing a number of partial data of the Monte-Carlo calculation performed by

(10) M. Baldo-Ceolin, C. Dilworth, W. F. Fry, W. D. B. Greening, H. Huzita, S. Limentani and A. E. Sichirollo: preprint and also reported at the *Padua-Venice Conference* (1957).

DE SABBATA, MANARESI and PUPPI for the π^--meson capture. The number of « events » on which our calculation is based is as follows: 435 for $\sigma_{\Lambda N} = 0$, 217 for $\sigma_{\Lambda N} = 15$ mb and 109 for each of the cases of $\sigma_{\Lambda N} = 30$ mb and 60 mb. In Fig. 2 the points linked by lines give the calculated fraction of capture stars with a given number of prongs. In Fig. 3 is shown the distribution of the total visible energy plus the binding energy of the capture stars with at least one prong.

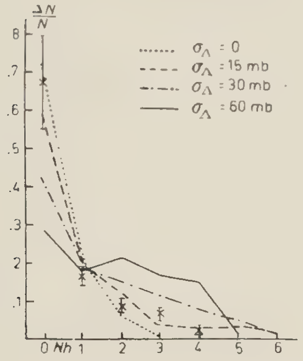

Fig. 2. – Prong distribution of Σ^- capture stars for different values of the Λ-N cross-section.

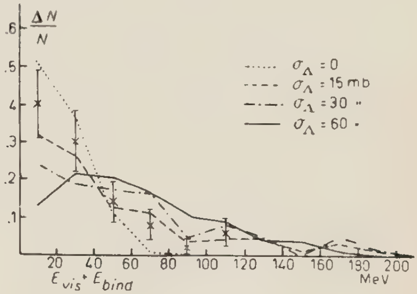

Fig. 3. – Visible energy release of Σ^- capture stars for different values of $\sigma_{\Lambda N}$.

The fraction of Λ-particles trapped in the nucleus for the four cases of Fig. 2 and 3 are found to be 0.0, 0.20, 0.42 and 0.58.

3. – Comparison with experiment.

The results of this calculation have been compared with the data of these four nuclear emulsion groups:

University of California (Livermore group) [11];

Max-Planck Institut für Physik, Göttingen [12];

University of Wisconsin [13];

University of California, Berkeley (Barkas group) [14].

The first two groups have attempted a direct determination of the percentage of Σ^--capture stars in which no ionizing prongs are emitted [15]. Both

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(11) See ref. (3).

(12) W. ALLES, N. N. BISWAS, M. CECCARELLI and J. CRUSSARD: *Proc. Padua-Venice Conference* (1957).

(13) W. F. FRY, J. SCHNEPS, G. A. SNOW, M. S. SWAMI and D. C. WOLD: *Phys. Rev.*, **107**, 257 (1957).

(14) W. H. BARKAS: *VII Rochester Conference Report* (1957).

(15) See ref. (11) and (12) and also WHITE's report at the *VII Rochester Conf.* (1957).

results, although obtained with different methods agree well estimating that zero prong Σ^- capture stars should be at least half of the total. The compilation of these two statistics gives the corresponding point (extreme left point) of Fig. 2. For the other points of this figure we have included the data of the Wisconsin and Berkeley groups as well as some more data of the Göttingen group, of good statistical weight but reliable only for captures with at least one prong. The data correspond to a mixture of captures by heavy and light nuclei; the implications of this will be discussed in Sect. **4**. For the data of Fig. 3 we have instead attempted to reduce the contribution of captures by light nuclei by eliminating as usual those events showing a proton of less than 4 MeV or an α-particle of less than 9 MeV.

Fig. 2 and 3 indicate that a complete transparency of the nucleus to the Λ-particle can be excluded because then the occurrence of a number of stars with several ionizing prongs or with rather high total visible energy, which needs to be interpretated as due to the trapping of the Λ, could not be explained. On the other hand, a cross-section around « geometric value » $(\pi(\hbar/m_{\pi}c)^2 = 63\text{ mb})$ or higher would not lead to an agreement with the observed high percentage of small stars. The curve corresponding to $\sigma_{\Lambda N} = 15\text{ mb}$ seems to fit better the experimental points. For this case, as mentioned previously, we have also computed the visible energy distributions, shown in Fig. 4, for the decay of the Λ stimulated exclusively by a proton or by a neutron. Since these curves differ little, the uncertainty on the ratio between the two reactions is not likely to be critical for our calculation.

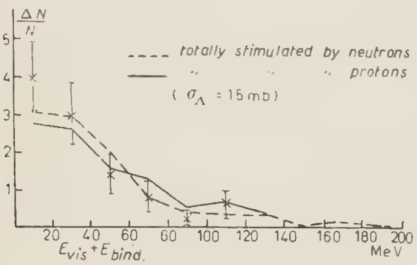

Fig. 4. – Visible energy release of Σ^- capture stars for Λ-decay stimulated exclusively by protons or by neutrons.

However, the relative frequency of the two reactions *e*) and *f*) should be substantially reflected in the energy spectrum of individual prongs of energy of more than about 40 MeV. Present statistics in this energy range is not sufficient for any conclusion.

4. – Effect of captures by light nuclei.

In the previous section the results of a calculation based on a model to be applicable to heavy nuclei have been compared with experimental data of nuclear emulsion which include also events in light nuclei. The elimination of these does not seem at present feasible for all classes of events; for instance for the zero prong stars.

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An α-particles model is probably a better picture for a light nucleus; it has been found for instance adequate to account for π^--captures in C, N and O ([16]). A calculation with this model ([17]) is however very involved and, as in case of π^- captures, is not likely to account fully for the experimental facts ([18]); however some qualitative features of the differences between Σ^--captures by light and heavy nuclei may be discussed.

As compared to the independent particle model in which the available energy is shared between two baryons, in an α-particle model this will be shared among five baryons so that the Λ will be emitted with a lower average energy (*). However due to the shorter nuclear path available, the probability of a Λ to loose energy by scattering will be less. These two effects will therefore play oppositely as far as the trapping probability of the Λ is concerned. Further, the cascading of particles in light nuclei will be less frequent so that the average excitation energy will be smaller. The occurence of charged prongs may however be not depressed since the Coulomb barrier is smaller.

These considerations indicate that the behaviour of captures by heavy nuclei might not be drastically different from those by light ones. For the latter case we may expect a lower percentage of zero prong capture stars and a corresponding increase of stars with low visible energy, in analogy with the case of π^--captures in which are involved energies of the same order as here.

5. – Concluding remarks.

In a recent paper Hornbostel and Zorn ([19]) have also obtained from the study of the trapping probability of Λ created inside the nucleus after the capture of a K^--meson an indication that the Λ-N scattering cross-section is significantly smaller than the geometric value. Their investigation deals with a rather complicated chain of reactions and with high excitation energy so that their results seem to be less quantitative than ours.

Assuming that at energies here considered the scattering is essentially isotropic, the Λ-N elastic cross-section may be derived from the values of the

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([16]) M. G. K. Menon, M. Muirhead and O. Rochat: *Phil. Mag.*, **41**, 583 (1950); P. Ammiraju and L. M. Lederman: *Nuovo Cimento*, **4**, 282 (1956).

([17]) A. C. Clark and S. N. Ruddlesden: *Proc. Phys. Soc.*, A **64**, 1064 (1951).

([18]) M. Demeur, A. Huleux and G. Vanderhaeghe: *Nuovo Cimento*, **4**, 509 (1956).

(*) Compare, for instance, the nucleon spectrum for π^--capture by heavy nuclei of Fig. 1 of ref. ([2]), having a maximum around 45 MeV (outside the nucleus), with that of captures by α-particles given in Figs. 2 and 3 of ref. ([17]), with maximum around 20 MeV.

([19]) J. Hornbostel and G. T. Zorn: preprint.

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volume integrals of the interaction potential, with the formula:

$$\sigma_{\Lambda N} = \frac{\mu_\Lambda^2}{\pi \hbar^4}\left[\frac{1}{4}(3U_p + U_a)\right]^2 .$$

This results from the Born approximation for a spherically symmetric square potential, but is also more generally valid [20]. Using the values of U_p and U_a given in Sect. 2, $\sigma_{\Lambda N}$ turns out to be 14.6 mb, in agreement with the value of about 15 mb, previously obtained.

* * *

We are most grateful to W. ALLES, who has worked with us at an early stage of this research for his contribution in discussions and suggestions.

We would further like to thank very deeply Prof. G. PUPPI and Dr. A. STANGHELLINI, as well as Drs. J. CRUSSARD, V. DE SABBATA, K. GOTTSTEIN, E. MANARESI, G. QUARENI and P. WALOSCHEK for valuable advices and criticisms. One of us (N.N.B.) is grateful to the Istituto di Fisica di Bologna for hospitality.

(20) H. BETHE and R. WILSON: *Phys. Rev.*, **83**, 690 (1951).

RIASSUNTO

Sulla base del modello a particelle indipendenti abbiamo cercato di interpretare i dati sperimentali sulla cattura di particelle Σ^- dai nuclei di emulsione fotografica. È risultato tra l'altro che la probabilità che le particelle Λ prodotte nella cattura rimangano nel nucleo, e quindi la dimensione media delle stelle di cattura dei Σ^-, dipende fortemente dal cammino libero medio delle particelle Λ in materia nucleare. Allo scopo di render conto dei dati sperimentali è opportuno assumere per la sezione di urto elastica Λ-nucleone, ad energie intorno ai 50 MeV, un valore di circa 15 mb.

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On the Connection of Spin with Statistics.

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(ricevuto il 20 Marzo 1958)

Summary. — The relation between spin and statistics is proved using only the most general assumptions.

The purpose of this note is to give a proof of Pauli's well known theorem on the connection of spin with statistics. The argument uses recently developed techniques of quantum field theory (1,2) and establishes the theorem in great generality. In particular, no assumptions are made about the form of the field equations or interactions (3). We show that if a relativistically invariant field theory has the properties:

a) No negative energy states;

b) The metric in Hilbert space is positive definite;

c) Distinct fields either commute or anticommute for space like separations,

then no field can have the « wrong » connection of spin with statistics.

Consider first a field Φ, transforming under some finite dimensional irreducible representation $\Lambda \to S(\Lambda)$ of the homogeneous Lorentz group (without

(1) A. S. Wightman: *Phys. Rev.*, **101**, 860 (1956).

(2) D. Hall and A. S. Wightman: *Dan. Vid. Selsk*, **31**, no. 5 (1957).

(3) After the present work was complete, I received a preprint of a paper on the same subject by G. Lüders and B. Zumino which deals with the case of spin zero and spin one half fields. For the case of neutral fields their result coincides with mine, but for charged fields they introduce a superfluos hypothesis of « gauge invariance ». The reader is referred to their paper for a description of previous papers on the same subject.

inversions). Then

$$U(0, \Lambda)\,\Phi(f)\,U(0, \Lambda)^{-1} = \Phi(f_\Lambda),$$

where

$$\Phi(f) = \int \mathrm{d}^4x f^\mu(x)\Phi_\mu(x)\,, \qquad f^\mu_\Lambda(x) = f^\lambda(\Lambda^{-1}x)S_{\lambda\mu}(\Lambda^{-1})\,,$$

and the f belong to an invariant space of test functions. $\{a, \Lambda\} \to U(a, \Lambda)$ is a unitary representation of the inhomogeneous Lorentz group. U operates on a Hilbert space of state vectors, containing a unique vacuum Ω_0. We define $\Phi^*(f) = [\Phi(\bar{f})]^*$, where $\bar{f}$ is the complex conjugate of f. The metric is $x^2 = x_0^2 - \boldsymbol{x}^2$.

The twofold vacuum expectation values are

$$F_{\mu\lambda}(\xi) = (\Omega_0, \Phi_\mu(x)\Phi^*_\lambda(y)\Omega_0)\,, \qquad G_{\mu\lambda}(\xi) = (\Omega_0, \Phi^*_\mu(x)\Phi_\lambda(y)\Omega_0)\,, \qquad \xi = x - y,$$

F and G can be extended to functions of a complex 4-vector $z = \xi - i\eta$, analytic when z^2 varies in the complex plane cut along the non-negative real axis (4). The region $\xi^2 < 0$ is within the cut plane, and in this region

$$G_{\mu\lambda}(-\xi) = \pm G_{\mu\lambda}(\xi)$$

the upper sign for integral spins (S single valued), the lower for half-integral spins (see Appendix for proof).

Using these results the « wrong » commutation relations lead to

$$F_{\mu\lambda}(\xi) + G_{\lambda\mu}(\xi) = 0 \qquad\qquad \xi^2 < 0$$

and by analyticity

$$F_{\mu\lambda} + G_{\lambda\mu} = 0 \ \text{ identically}\,.$$

Now consider the state vectors $\Phi(f)\Omega_0$ and $\Phi^*(\tilde{f})\Omega_0$, where $\tilde{f}(x) = \bar{f}(-x)$. We compute,

$$\|\Phi(f)\Omega_0\|^2 + \|\Phi^*(\tilde{f})\Omega_0\|^2 = \int \mathrm{d}^4x\,\mathrm{d}^4y \bar{f}^\mu(x)[G_{\mu\lambda}(x-y) + F_{\lambda\mu}(x-y)]f^\lambda(y) = 0\,.$$

(4) The argument given in (1) is for scalar fields but is easily extended to the case of arbitrary fields. See, for example, A. S. Wightman hectographed notes on lectures at Institute Henry Poincaré (1957). The same remarks apply to the result that a theory is uniquely determined by its vacuum expectation values.

Therefore $\Phi(f)\Omega_0 = \Phi^*(\tilde{f})\Omega_0 = 0$ for all f. Using assumption c) and the results of HALL and WIGHTMAN on the real analytic points we conclude that any vacuum expectation value containing Φ vanishes identically and consequently that such a field is zero [2 4].

* * *

The author is indebted to A. S. WIGHTMAN for his valuable comments and advice. He also thanks Professor NIELS BOHR for the hospitality extended to him at the Institute for Theoretical Physics.

APPENDIX [5]

G satisfies the relation,

$$S_{\mu\mu'}(\Lambda)S_{\lambda\lambda'}(\Lambda)G_{\mu'\lambda'}(\xi) = G_{\mu\lambda}(\Lambda\xi) ,$$

by standard invariance arguments one finds that G is given uniquely as a sum of terms of the form

$$\Gamma_{\mu\lambda}(\alpha_1 \dots \alpha_n)\xi_{\alpha_1} \dots \xi_{\alpha_n} g(\xi) , \qquad (\alpha_i = 1 \dots 4),$$

where

$$\bar{S}_{\mu\mu'}(\Lambda)S_{\lambda\lambda'}(\Lambda)\Gamma_{\mu'\lambda'}(\alpha_1 \dots \alpha_n) = \Gamma_{\mu\lambda}(\alpha_1' \dots \alpha_n')\Lambda_{\alpha_1'\alpha_1} \dots \Lambda_{\alpha_n'\alpha_n} , \qquad g(\Lambda\xi) = g(\xi) ,$$

and n is always even for S single valued, and always odd for S double valued.

On extending G to complex values, it becomes

$$G_{\mu\lambda}(z) = \dots + \Gamma_{\mu\lambda}(\alpha_1 \dots \alpha_n) z_{\alpha_1} \dots z_{\alpha_n} g(z) + \dots ,$$

where $g(z) = g(\Lambda_c z)$ for z in the cut plane, and Λ_c belongs to the complex Lorentz group (with determinant $+1$). In particular $g(\xi) = g(-\xi)$ for $\xi^2 < 0$. This proves the original statement.

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[5] The property proved in this appendix follows from the analysis of W. PAULI: *Phys. Rev.*, **58**, 716 (1940).

RIASSUNTO (*)

Usufruendo solo delle ipotesi più generali si dimostra la relazione tra spin e statistica.

(*) *Traduzione a cura della Redazione.*

Pion Multiplicity in Antinucleon Annihilation (*).

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Centro Siciliano di Fisica Nucleare - Catania

(ricevuto il 26 Marzo 1958)

Summary. — Pion multiplicity in antinucleon annihilation has been calculated using Dyson's hypothesis of pion-pion resonance interaction. The percentage of K-mesons was also evaluated supposing that their volume of interaction has a radius given by the Compton wavelength of K-mesons. Results are in reasonable agreement with experimental data.

It is well known that the application of Fermi's statistical theory (1) to the problem of pion production in antinucleon annihilation gives (2) for pions a multiplicity $n_\pi = 3, 4$ that is not in agreement with experimental results (3) $\bar{n}_\pi = 4.7 \pm 0.4$.

For obtaining such a high multiplicity, with Fermi's theory, one should take a radius of interaction of the order of 2 or 3 times the π-meson Compton wavelength; but this hypothesis is not very reliable.

Many efforts have been made in attempting to modify Fermi's theory efficiently (+) (4); most of these were unsuccessful or not well justified.

(*) A preliminary report on the present investigation was communicated by M. Cini to the *International Conference of Padua-Venice* (September 1957).

(+) In a recent work of Z. Koba and G. Takeda (not yet published) the authors succeeded in explaining the high pion multiplicity; they suppose that annihilation takes place only between the cores of nucleons; the pions resulting from the annihilation have to be added to those coming from the virtual clouds wich have lost their centre to associate with. But we remark that the authors have derived the energy released in the core annihilation from the mean experimental energy of the pions that come out from the entire process, thus assuming in their phenomenological treatment an experimental result that they should explain.

(1) E. Fermi: *Progr. Theor. Phys.*, **5**, 570 (1950); *Phys. Rev.*, **92**, 452 (1953).

(2) R. E. Marshak: *VII Conference of Rochester*, chap. X (1957), p. 27.

Our attempt is to explain the high pion multiplicity by using Dyson's hypothesis ([5]) introduced for justifying the second maximum in π-p scattering cross-section. Following DYSON we suppose the existence of a π-π resonant interaction in the state of total isotopic spin $T = 0$ at a relative momentum of 250 MeV/c and $l = 0$. We must observe that this hypothesis is not too well defined, but we noticed, in making our calculations, that statistical weights for various final states were not too much sensitive to variations of the relative momentum.

For introducing this resonance effect in the frame of statistical theory, we have adapted to our problem a method suggested by LANDAU and developed by BELEN'KJI ([6]) (see Appendix).

Numerical computations of statistical weights were made by means of saddle point approximation, that is exactly relativistic ([7]).

In our calculations we took in account conservation of total isotopic spin, and assumed for interaction radius the Compton wavelength of π-mesons.

In Table I the statistical weights $S(n_\pi)$ are reported for various final states (*).

TABLE I. – *Statistical weights for various final states* (K-*meson production is not considered*).

n_π	$S(n_\pi)$
2	3.10%
3	18.06%
4	34.85%
5	24.29%
6	19.7 % (+)

$\bar{n}_\pi = 4.4$

(+) The value here reported for $S(6_\pi)$ is not exactly calculated; this approximation does not infirm the conclusions of this work.

([3]) W. H. BARKAS, R. W. BIRGE, W. W. CHUPP, A. G. EKSPONG, G. GOLDHABER, S. GOLDHABER, H. H. HECKMAN, D. H. PERKINS, J. SANDWEISS, E. SEGRÈ, F. M. SMITH, L. VAN ROSSUM, E. AMALDI, G. BARONI, C. CASTAGNOLI, C. FRANZINETTI and A. MANFREDINI: *Phys. Rev.*, **105**, 1037 (1957); A. G. EKSPONG: *VII Rochester Conference*, chap. X (1957), p. 18.

([4]) I. I. POMERANČUK: *Dokl. Akad. Nauk*, **78**, 88 (1951); E. C. G. SUDARSHAN: *Phys. Rev.*, **103**, 777 (1957); R. E. MARSHAK: *VII Rochester Conference*, chap. X (1957), p. 27.

([5]) F. J. DYSON: *Phys. Rev.*, **99**, 1037 (1957).

([6]) L. LANDAU: *Statistical Physics*, Sect. **75** (Gostehizdat, 1951); S. Z. BELEN'KJI: *Nucl. Phys.*, **3**, 259 (1956).

(*) All the statistical weights reported in this work are calculated for $\bar{p}p$ (or $\bar{n}n$) annihilations; the results for $\bar{p}n$ (or $\bar{n}p$) annihilations are substantially the same.

So we obtain the value $\bar{n}_\pi = 4.4$; this value is not in disagreement with the experimental results.

Another experimental feature of annihilations processes is the low percentage of K-mesons. In this respect we can make two hypotheses:

a) The radius of interaction for K-mesons is given by the Compton wavelength of the π-mesons.

b) The radius of interaction for K-mesons is given by the Compton wavelength of the K-mesons themselves (*).

For case *a*) the results are reported in Table II (+); for case *b*) the results are reported in Table III (+).

TABLE II.

n_π	n_K	$S(n_\pi, n_K)$
2	0	2.37%
3	0	13.76%
4	0	26.55%
5	0	18.50%
6	0	15 % (+)
0	2	2.04%
1	2	9.18%
2	2	10 % (+)
3	2	2.6 % (+)

(+) See note(+) in Table I

Percentage of cases in which a couple of K-mesons appears: $\sim 24\%$.

Pion multiplicity: $\bar{n}_\pi = 3.8\%$.

Statistical weights for various final states.

Interaction radius $= \hbar/m_\pi c$.

TABLE III.

n_π	n_K	$S(n_\pi, n_K)$
2	0	3.09%
3	0	17.93%
4	0	34.61%
5	0	24.12%
6	0	19.55% (+)
0	2	0.06%
1	2	0.18%
2	2	0.28% (+)
3	3	0.18% (+)

Percentage of cases in which a couple of K-mesons appears: 0.7%.

Pion multiplicity: $\bar{n}_\pi = 4.4$.

Statistical weights for various final states

Interaction radius for π-mesons $= \hbar/m_\pi c$.

Interaction radius for K-mesons $= \hbar/m_K c$.

Experimental results give (7) a value less than $\sim 6\%$ for the percentage of cases in which a couple of K-mesons appears.

We can see that the case *b*) is the most reliable (the discrepancy between the value here calculated and the experimental one is not really significant because one can assume for the radius of interaction for K-mesons a slightly larger value than the K-mesons wavelength.

(*) This hypothesis has been introduced by BARAŠENKOV *et al.* (9).

(+) We took in account strangeness conservation, that is associated production.

(7) G. E. A. FIALHO: *Phys. Rev.*, **105**, 328 (1957).

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In conclusion we can say that, taking in account a π-π resonant interaction (for which there is already some evidence), the statistical theory gives good results for the pion multiplicity, and that the low percentage of K-mesons can be explained by assuming the hypothesis already introduced in the study of nucleon-nucleon collisions [8], namely that their volume of interaction is much smaller than the one of π-mesons.

* * *

I must expressely thank Prof. M. Cini for his continuous encouragement ad Dr. A. Agodi and B. Vitale for their interest in this work.

Appendix

The formula we used for evaluating the statistical weights of Table I is the following [6]:

$$S(n_\pi)=K_{n,T}[S_0(n_\pi)+K_1(2l+1)S_0((n-2)_\pi,\,1W_0)+K_2(2l+1)^2S_0((n-4)_\pi,\,2W_0)+...].$$

Here $S(n_\pi)$ is the statistical weight for a final state in wich n π-mesons are present; $S_0(n_\pi)$, $S_0((n-2)_\pi,\,1W_0)$, ..., are the statistical weights evaluated in absence of interaction (and not comprehensive of the coefficient that takes into account the conservation of total isotopic spin; this coefficient ($K_{n,T}$) is the same for all the S_0, for a given multiplicity; the various $K_{n,T}$ have been calculated by Yeivin-De Shalit [10]), in the cases respectively, in which no couple of interacting pions is present, one couple of interacting pions is present, and so on.

These statistical weights have to be calculated as if in the final state were present: n pions, $(n-2)$ pions and a « particle » W_0, $(n-4)$ pions and two « particles » W_0, and so on.

W_0 has to be conceived, formally, as a particle of mass:

$$M_{W_0}=2m_\pi+E_0\,,$$

where E_0 is the kinetic energy of the two pions at resonance in the c.m. system.

The coefficients K_1, K_2, ... take into account the various modes of selecting a couple (or two, three, ...) of interacting pions and also the fact that such a couple interacts only in a state of total isotopic spin $T=0$.

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(8) A. G. Ekspong: *VII Rochester Conference*, chap. X, p. 18.

(9) V. S. Barašenkov, B. M. Barbašev, E. G. Bubelov and V. M. Maksimenko: *Nucl. Phys.*, **5**, 17 (1958).

(10) Y. Yeivin and A. De Shalit: *Nuovo Cimento*, **1**, 1146 (1955).

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In to order evaluate these coefficients we used those calculated by NIKISHOV [11] together with obvious combinatorial methods; the results are collected in the following table:

n_π	K_1	K_2
3	1.299	—
4	2.886	1.443
5	4.120	2.580

Statistical weights reported in Table II are evaluated in a similar way.

With regard to statistical weights of Table III, they are different from those of Table II only in the space weight factors of the various S_0; we remember that the space weight factor, in the case of Table II, has the following form:

$$\frac{1}{G}\left[\frac{\Omega}{(2\pi\hbar)^3}\right]^{n-1}, \tag{1}$$

where G is the well known factor taking into account identity of particles and Ω is the interaction volume.

In the case considered in Table III (the interaction volume of K-mesons is different from that of π-mesons), (1) is replaced by [9] an average space weight factor:

$$\frac{nV_{\mathrm{K}}^{k}V_{\pi}^{n-1}+kV_{\mathrm{K}}^{k-1}V_{\pi}^{n}}{n+k}=\frac{1}{n+k}\,[n\xi+k]\,\xi^{k-1}\,V_{\pi}^{n+k-1}\,,$$

where n is the number of pions, k is the number of K-mesons, V_π and V_{K} are the space weight factors (1) for pions and K-mesons respectively, and $\xi=V_{\mathrm{K}}/V_\pi=(m_\pi/m_{\mathrm{K}})^3$.

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[11] A. I. NIKISHOV: *Journ. Exp. Theor. Phys.*, **3**, 976 (1956, American Edition).

RIASSUNTO

Si è calcolata la molteplicità pionica nell'annichilazione di un antinucleone ed un nucleone introducendo un'ipotesi proposta da DYSON riguardante una interazione di risonanza pione-pione. Si è calcolata inoltre la percentuale di mesoni K prodotti supponendo che il raggio del loro volume d'interazione sia dato dalla lunghezza d'onda Compton del mesone K stesso. I risultati così ottenuti sono in buon accordo con quelli sperimentali.

The Use of the Neon Flash-Tube for the Precise Location of Particle Trajectories.

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Department of Physics, The Durham Colleges in the University of Durham

(ricevuto il 18 Gennaio 1958)

Summary. — The passage of fast cosmic ray particles through an array of neon flash-tubes has been studied and the accuracy of location of the particle trajectories has been found to be as high as in conventional cloud chambers.

1. – Introduction.

A new technique for the detection of ionizing particles has been reported by Conversi *et al.* (1) and further developments have been made by Barsanti *et al.* (2) and Gardener *et al.* (3). The basic detector is a glass tube filled with neon which, under certain conditions, can be made to glow after traversal by an ionizing particle. It has been pointed out by Gardener *et al.* that such a detector should make a satisfactory particle-locator in a magnetic spectrograph. In particular, the device should be invaluable in cosmic ray studies where detecting layers having large areas of particle collection and high spatial resolution are required.

Before such a new technique can be used in this way a detailed study of various aspects is necessary: this paper describes such a study.

(1) M. Conversi and A. Gozzini: *Nuovo Cimento*, **2**, 189 (1955).

(2) G. Barsanti, M. Conversi, S. Focardi, G. P. Murtas, C. Rubbia and G. Torelli: *Proceedings of C.E.R.N. Symposium*, II, 56 (1956).

(3) M. Gardener, S. Kisdnasamy, E. Rössle and A. W. Wolfendale: *Proc. Phys. Soc.*, B **70**, 687 (1957).

2. – Experimental arrangement.

In constructing a detecting array the first problem is the optimum tube diameter to choose. For tubes having 100% internal detection efficiency and infinitesimal wall thickness, the number of tubes required to cover a given area and produce a given spatial resolution is virtually constant, viz: 6 layers of tubes of diameter d give nearly the same resolution as 2 layers of tubes of diameter $d/3$. However, considerations of the gas pressure required to give a reasonable number of ion-pairs and the difficulties involved in manipulating tubes having very thin walls dictate a tube diameter of several mm. The tubes used in the experiments to be described had internal diameter 5.9 mm and external diameter 7.7 mm.

The apparatus used to study the accuracy of location of particle trajectories is shown in Fig. 1. Four tube arrays, A, B, C and D, and associated Geiger counters served to trace the trajectories of high energy particles at sea level. The tubes were rigidly mounted 8.0 mm apart between parallel electrodes: scattering was reduced by replacing the central regions of the electrodes by Aluminium foil. The tubes were filled with commercial neon at a pressure of 65 cm Hg and the residual air pressure was estimated to be 10^{-3} mm Hg. The characteristics of the high voltage pulse applied across the plates after a 4-fold coincidence were as follows: peak height 6.3 kV/cm, time delay to the start of the pulse 2.0 μs, rise-time to 75% of the peak height 0.5 μs, pulse length 4.0 μs. Under these conditions the internal efficiency of the tubes was 70% and the layer efficiency 52%. In order to reduce the scattering of particles in the layers a low momentum cut-off was imposed by demanding further penetration of a lead block below the apparatus. When operated as a spectrograph a magnet will be inserted between arrays B and C and the incident and emergent directions determined by AB and CD respectively. In the experiment reported here the magnet was removed but the angle, θ, between the apparent trajectories in AB and CD was measured; the distribution of this angle after correction for scattering gave the noise-level or « no-field » limit of measurement.

Before proceeding it is necessary to consider the optimum geometrical arrangement of the individual layers of tubes within an array. If the tubes are arranged vertically one above the other it is possible for vertical particles to pass through the dead space between the sensitive regions of adjacent tubes in each layer with consequent loss in overall efficiency of detection, since events where no tube flashes in one array cannot be used. The overall efficiency is defined as the ratio of the number of measurable photographs to the total number taken. Similarly a vertical particle passing through the sensitive region of a tube in each layer can give rise to considerable uncertainty in the location of the trajectory. A staggering angle can be chosen to minimize these effects if the angular distribution of the particles is known and this has been done here. The practical performance of this arrangement has been compared with that for vertical arrays and whereas the overall efficiency of particle detection is higher the increase in precision of location in the former case is not very great.

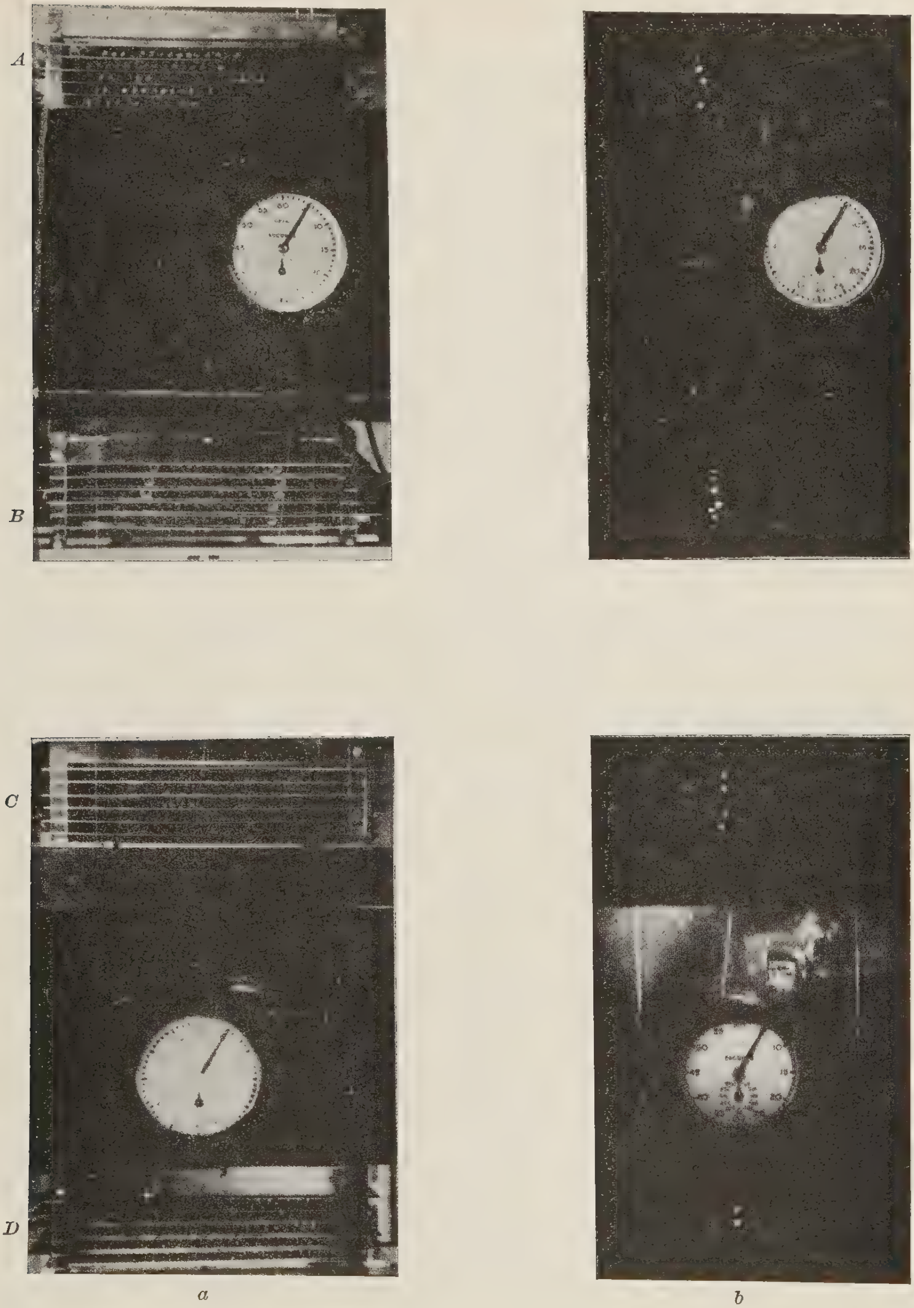

Fig. 1. – The experimental arrangement. (*a*) Shows the arrangement of 4 tube arrays (A, B, C, D) and the cotton fibres used to align the arrays (A, B and C, D are photographed by separate cameras). (*b*) Shows the passage of a single fast μ-meson through the apparatus.

3. – Measurements of the trajectories.

Measurements of the track directions in AB and CD were made by 3 methods:

(i) The centre of gravity of the array formed by the centres of the tubes which flashed at each level was calculated and the inclination of the line joining the appropriate pairs was determined.

(ii) A full-scale diagram of the tube assembly was drawn and a cotton fibre adjusted over its surface to give the best estimate of the track direction.

(iii) The photographic records were projected on to a rotable screen ruled with close parallel lines and the direction of a line determined which satisfied the condition that the sum of the distances from the centres of the flash-tube images to the line was a minimum.

In each method where a tube was observed to flash in such a position as to be incompatible with the trajectory defined by the other tubes this flash was rejected. Several comments can be made on these methods. Method (i) has the advantage of being completely objective but some information present in the record is not used, that is the reduction in uncertainty of position at one level arising from the knowledge of the approximate direction given by the information from the two levels taken together. In principle this can be overcome by applying the method of least squares, say, to the data from the two levels in turn, but the calculations are tedious.

Method (ii) is subjective but has the advantage that not only the excited tubes, but also the gaps between the sensitive regions, can be used to locate the trajectories.

Method (iii) on the other hand relies only on the excited tubes since these are the only images visible on the film. A basic assumption is that the image accurately represents the end of the tube, and with carefully constructed tubes, this condition is attained.

All the photographs taken in this experiment have been measured using methods (ii) and (iii) and a smaller number with method (i). The conclusion is that method (i) is inferior to the other two and that there is little difference between the results of methods (ii) and (iii); method (iii) is quicker to apply and is indicated for applications where a high particle rate is encountered. Where great significance is attached to individual particles, checking by method (ii) is desirable. A further check can be carried out by determining the points of intersection with the central plane of the trajectories found from the upper and lower halves of the apparatus. The separation of these points will be small for single particles.

The distribution in θ found from measurements with the present apparatus has been found to be $0.33° \pm 0.02°$. This distribution arises from the following causes:

a) The scattering of particles in arrays B and C.

b) The error of setting the measuring device (cross-wire etc.) on the best estimate of the trajectory.

c) The error of interpretation, i.e. the difference between the estimated and true trajectories.

d) The traversal of the apparatus by more than one particle.

Of these causes only *c*) and *d*) need further amplification. Item *c*) refers to cases where one or more of the tubes discharged were not traversed by the ionizing particle itself. Probably the most important contribution arises from unrecognized knock-on electrons, e.g. a case where the fast particle passes through the insensitive region of one tube and an associated knock-on electron passes through the next tube such that a systematic difference between the best estimate and the true trajectory cannot be recognised. Also included in *c*) is the spread of true trajectories about those best-estimates derived using the criterion of minimum sum of the distances from estimated trajectory to the centres of the excited tubes.

The contribution from scattering and errors of setting are known and thus the error from the combination of *c*) and *d*) can be determined. Writing σ_{sc}, σ_s, σ_{rem} for the r.m.s. deflections due to scattering, setting, and the remaining causes, we have

$$\sigma_{sc} = 0.22^\circ , \qquad \sigma_s = 0.14^\circ \qquad \text{giving} \qquad \sigma_{rem} = 0.20^\circ \pm 0.03^\circ .$$

It is usual to quote the uncertainty as that of position at a single level (e.g. the middle of each array) and this can be calculated. It must be borne in mind however that this uncertainty is less than would be observed if the single level alone were observed on account of the information on direction given by the combined data from the two levels.

The required r.m.s. uncertainty, δ, is given by $\delta = L\sigma_{rem}/2$ where L is the separation of each pair of measuring levels. The result is

$$\delta = \underline{(0.62 \pm 0.10)\ \text{mm}}\ .$$

This value can in principle be compared with that expected from the geometry of the arrangement and the efficiency characteristics of single tubes. Limiting forms of the efficiency variation across the tube diameter have been considered and the value of δ determined graphically in each case. The result is

$$\underline{\delta_1 = 0.46\ \text{mm}}\ ,$$

for an efficiency function $\eta = f(r/a)$ where a is the radius and r is the distance from the centre, with $f(r/a) = 1.0$ for $r/a < 0.70$ and $f(r/a) = 0$ for $r/a > 0.70$ and

$$\delta_2 - 0.75\ \text{mm}\ ,$$

for $f(r/a) = 0.70$ for all values of $r/a \leqslant 1$.

The actual form of the efficiency variation across the tubes was studied in a separate experiment in which the arrangement of arrays was modified. Array A was placed mid-way between arrays B and C and tracks were defined by the observed flashes in B and C. The distances of these tracks from the

centres of those tubes in A through which the particles had passed were accurately measured. The resulting variation of internal efficiency with distance from the centre is shown in Fig. 2. Also shown is the expected variation according to a theoretical analysis of LLOYD (to be published) where normalization to the same internal efficiency has been made. A broadening of the expected distribution is necessary due to the uncertainty in track location (~ 0.3 mm) but this is too small to be significant. It is apparent that the agreement with theory is good.

Since the form of the efficiency variation is roughly mid-way between the two extreme assumptions mentioned above it is reasonable to expect that the resulting uncertainty be mid-way between δ_1 and δ_2. The observed value, of 0.64 mm, is thus in good agreement with expectation.

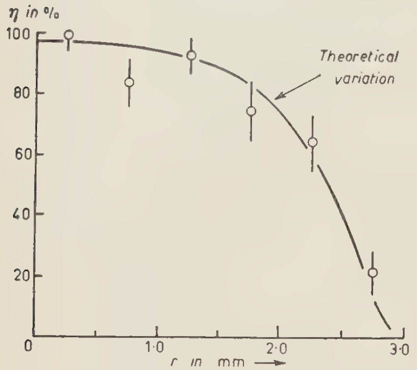

Fig. 2. – The variation of internal efficiency with distance of the ionizing particle from the centre of the tube. η is the internal efficiency and r the distance from the tube centre. The theoretical variation allows for the loss of electrons by diffusion to the walls in the time delay occurring between the passage of the particle and the application of the high voltage pulse.

4. – Conclusions.

It can be concluded from the discussion of the accuracy of location that the effect of spurious phenomena such as discharges produced by unrecognized knock-on electrons, is small.

A comparison can be made between this method and the cloud chamber. In a cloud chamber the inherent track width, for cosmic ray studies, is ~ 1 mm. It is possible to determine the centre of a track to within about 0.2 mm but to this uncertainty must be added that arising from gaseous distortion. By careful temperature control distortion can be reduced but both long and short term variations in distortion produce difficulties. In the Manchester high energy spectrograph (HOLMES *et al.* [4]) cloud chambers were used as detectors and the overall uncertainty in track location was shown, in a subsidiary experiment by LLOYD *et al.* [5] to be about 0.8 mm. Further reduction by more careful temperature control is presumably possible but the location uncertainty in a cloud chamber is inherently variable and in that respect alone is inferior to the flash-tube array with its constant geometrical uncertainty.

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(4) J. E. R. HOLMES, B. G. OWEN, A. L. RODGERS and J. G. WILSON: *Proc. Phys. Soc.*, A **68**, 793 (1955).

(5) J. L. LLOYD, E. RÖSSLE and A. W. WOLFENDALE: *Proc. Phys. Soc.*, A **70**, 421 (1957).

It is apparent that a further reduction in error of location can be made in a flash-tube array by increasing the number of layers, an upper limit being set by the maximum tolerable scattering uncertainty. In cosmic ray experiments, however, the limit is liable to be set more by the lack of a sufficient flux of particles of small magnetic deflection than the difficulty in measuring the deflections.

* * *

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RIASSUNTO (*)

È stato studiato il passaggio di radiazione cosmica veloce attraverso un rivelatore costruito con scintillatori al neon e l'esattezza dell'individuazione delle traiettorie delle particelle è risultata altrettanto buona di quella permessa dalle camere a nebbia convenzionali.

(*) *Traduzione a cura della Redazione.*

Study of τ^+-Meson Decay in a Propane Bubble Chamber (*).

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(ricevuto il 22 Marzo 1958)

Previous large sample studies of τ^+-meson decay events have been carried out by means of the nuclear emulsion technique exclusively ([1]). There is, of course, always the possibility of a geometric bias existing in experiments of this sort and therefore it is of interest to examine data which have been collected and analyzed by a different experimental method.

In a recent exposure of the University of Michigan propane bubble chamber to a K^+-meson beam at the Brookhaven Cosmotron, 136 τ^+-mesons were observed to decay from rest. The details of the method of collection and processing of the data have been considered elsewhere and will not be discussed here ([2]). The purpose of this note is to present, briefly, the results of the analysis of the τ^+ decay events.

For each event the ranges of the stopping pions were measured. In addition, the angles included between the pion tracks were also obtained. This information, of course, overdetermines the event kinematically, but does have the advantage that it allows a check on the internal consistency of the experimental method. As a further check over a hundred μ^+-meson tracks arising from π's decaying from rest were also measured. The average μ^+ length obtained from these measurements was found to be in good agreement with the range predicted by the range energy tables calculated for propane (C_3H_8).

For 118 of the decays the energies and the charge of the three pions was unambiguously determined. The remaining 18 events were such that the π^--meson and one of the π^+-mesons escaped from the chamber. Thus, in these cases the charge of the two escaping particles is ambiguous. It is to be noted that it is only the charge

(*) Supported in part by the U. S. Atomic Energy Commission.

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([1]) *Proceedings of the Seventh Annual Rochester Conference*, **7**, 19 (1957).

([2]) D. I. Meyer, M. L. Perl and D. A. Glaser: *Phys. Rev.*, **107**, 279 (1957).

TABLE I. – *The logarithms to the base* 10 *of the probability ratios are given for the* 1^+, 1^- and 2^+ *spin-parity combinations compared to the* 0^- *spin-parity combination, as calculated by the Dalitz-Fabri analysis. For the expected values the standard deviations are given.*

	$\log_{10} P1^+/P0^-$	$\log_{10} P1^-/P0^-$	$\log_{10} P2^+/P0^-$
Observed value if 18 ambiguous events are distributed as 0^- .	-13	-28	-11
Observed value if 18 ambiguous events are distributed as other spin-parity combination. . .	-12	-23	-8
Expected value if 0^- is correct	-12 ± 4	-53 ± 6	-49 ± 13
Expected value if other spin-parity combination is correct	$+8 \pm 2$	$+133 \pm 30$	$+39 \pm 8$

which is ambiguous since the energies of the escaping particles are known from the angular measurements.

The energy distribution, Fig. 1, and the $\cos\theta$ distribution, Fig. 2, are the same within statistics as the distributions derived from emulsion measurements [1]. Therefore no systematic bias in the emulsion method has been disclosed by the bubble chamber method.

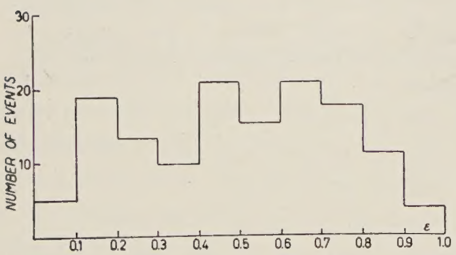

Fig. 1. – Distribution of the events versus the energy of the π^--meson. ε is the kinetic energy of the π^--meson divided by the maximum kinetic energy of the π^--meson.

The Dalitz-Fabri analysis of the data supports the 0^- spin-parity combination, as can be seen from Table I which presents the probability ratios. The agreement of the probability ratios with the 0^- spin-parity combination cannot

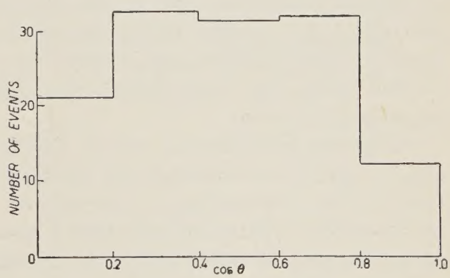

Fig. 2. – Distribution of the events versus $\cos\theta$. θ is the angle in the center of mass system between the momentum vector of the π^--meson and the relative momentum vector of the two π^+-mesons.

be altered by distributing the 18 ambiguous events in the most unfavorable way. A χ^2 test on the energy spectrum yields a value of $P = 25\%$ when compared to the 0^- distribution function and for the angular distribution a value of $P = 7\%$ is obtained. A complete list of the events can be obtained from the first author (T.F.Z.).

LIBRI RICEVUTI E RECENSIONI

H. E. DUCKWORTH - *Mass spectroscopy.* Cambridge Univ. Press., 1958, pp. XVI - 206.

Il libro, che fa parte dei « Cambridge Monographs on Physics », riassume in poco meno di 200 pagine l'attuale situazione nel campo della spettrometria di massa e delle sue principali applicazioni.

L'esposizione del DUCKWORTH, chiara ed ordinata, illustra concisamente i progressi compiuti e i dati acquisiti. I problemi fisici sono trattati esaurientemente, ma senza entrare nei dettagli, in modo da rendere il libro di facile lettura al non specialista; in questo stesso spirito le questioni più tecniche, come l'elettronica e l'alto vuoto, sono state evitate per non appesantire il libro.

Dopo una introduzione storica, cinque capitoli del libro sono dedicati all'ottica ionica e alla strumentazione, due alla determinazione rispettivamente delle masse atomiche e delle abbondanze isotopiche e tre alle applicazioni. Per quanto riguarda queste ultime, solo quelle riguardanti la fisica e la geologia sono trattate in qualche dettaglio: un intero capitolo concerne le applicazioni in fisica nucleare, e relativamente ampio spazio è dedicato alla ionizzazione e alla dissociazione delle molecole soggetto ad urto da parte di elettroni, e allo studio degli ioni metastabili; infine sono trattate le applicazioni alla geologia, sia quelle riguardanti gli isotopi radiogenici (determinazione delle età geologiche con i metodi U-Pb A-K e Rb-Sr), sia quelle riguardanti gli isotopi stabili (variazioni delle abbondanze in conseguenza di processi chimico-fisici occorrenti in natura). Vengono solo sfiorati i problemi in cui lo spettrometro di massa funziona esclusivamente come strumento analitico, per esempio l'uso degli isotopi per lo studio del meccanismo delle reazioni chimiche e i metodi di separazione degli isotopi.

Il libro è di qualche interesse anche per lo specialista; in particolare la bibliografia è molto ampia e, per quanto riguarda la strumentazione, pressochè completa.

G. BOATO

Direttore responsabile: G. POLVANI

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